physics UK
Smarter practice. Stronger understanding

SOLUTIONS
ATTEMPT CHECK CORRECT IMPROVE

OCR A A Level Simple Harmonic Motion 2 solutions

Matching solutions and teaching prompts for the class review

Specification points covered

- 5.3.1(e) Derive velocity for SHM as $v = \pm\omega\sqrt{(A^2 - x^2)}$ and identify $v_max = \omega A$.
- 5.3.1(f) Recognise that the period of a simple harmonic oscillator is independent of amplitude (isochronous behaviour).
- 5.3.1(g) Use graphical methods to relate displacement, velocity and acceleration during SHM.
- 5.3.2(a) Describe the interchange between kinetic and potential energy during SHM.
- 5.3.2(b) Interpret energy-displacement graphs for a simple harmonic oscillator.

Teaching focus

Make pupils use gradients and directions, not memorised position-only statements. A piston can have the same displacement with either velocity sign.
Use the free calibration only for conserved mechanical energy. In pump operation the motor supplies energy and the water and resistive processes remove it.

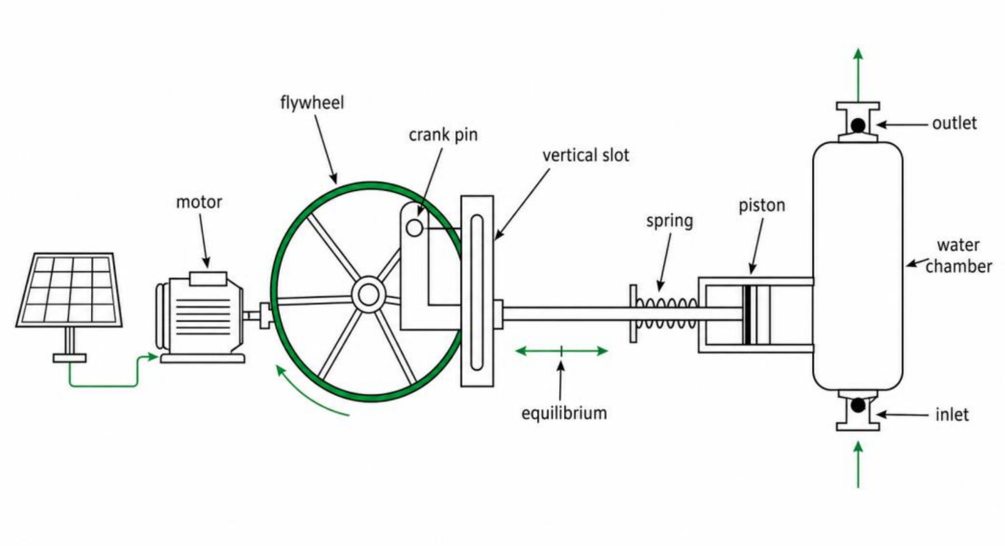

Figure 1: solar-powered Scotch-yoke water pump and calibration spring

1. Reading and modelling the oscillation

1. During normal operation the pump flywheel rotates uniformly at 75.0 rev min^{-1}. One revolution produces one complete piston oscillation. Calculate the oscillation frequency f, period T and angular frequency ω. [3 marks]

Answer / mark points: $f = 75.0/60 = 1.25$ Hz. $T = 1/f = 0.800$ s. $\omega = 2\pi f = 2\pi \times 1.25 = 7.85$ rad s^{-1}.
Teacher prompt / common error: Convert revolutions per minute to hertz before using $\omega = 2\pi f$. Frequency and angular frequency are different quantities with different units.

2. For a low-friction calibration run, the motor and water chamber are disconnected. The piston-spring assembly is released from rest at x = +40.0 mm when t = 0. Using your angular frequency from Question 1, write an equation for x in terms of t and calculate x at t = 0.100 s. [3 marks]

Answer / mark points: The piston starts at its positive maximum displacement, so a cosine model has the correct initial condition: $x = 0.0400 \cos(7.85t)$, with x in metres and t in seconds. At $t = 0.100$ s, $x = 0.0400 \cos(7.85 \times 0.100) = +2.83 \times 10^{-2}$ m.
Teacher prompt / common error: Cosine follows from $x(0) = +A$ and $v(0) = 0$. Calculator angles must be in radians.

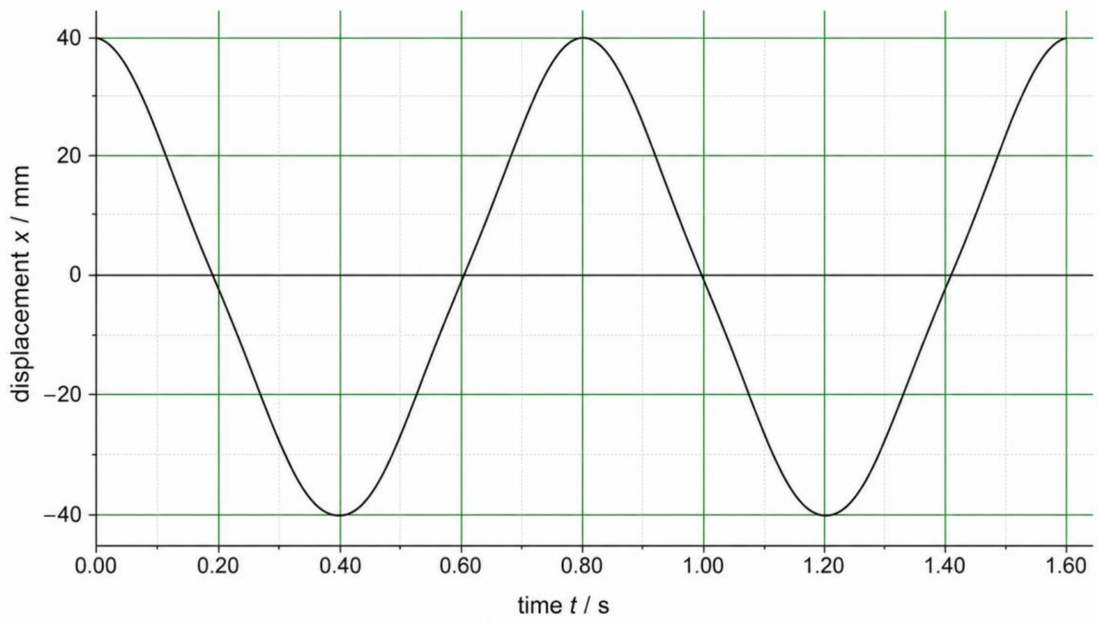

Figure 2: amplitude 40.0 mm; period 0.800 s

3. Figure 2 is the displacement-time record from the calibration run. (a) Determine the amplitude and period. (b) The positive direction is to the right. Use the gradient of the graph to state the direction of the piston velocity at t = 0.300 s. [3 marks]

Answer / mark points: (a) $A = 40.0$ mm $= 0.0400$ m and $T = 0.800$ s. (b) At 0.300 s the graph has a negative gradient, so $v = dx/dt$ is negative and the piston is moving to the left.
Teacher prompt / common error: The gradient of an x-t graph is velocity. Say that velocity is negative or the piston moves left; do not say the displacement is accelerating.

2. Maximum values and signed velocity

4. Calculate the maximum speed and the maximum magnitude of the acceleration of the piston. State the position or positions at which each maximum occurs. [4 marks]

Answer / mark points: $v_{max} = \omega A = 7.85 \times 0.0400 = 0.314$ m s^{-1}; this occurs at equilibrium, $x = 0$. $a_{max} = \omega^2 A = 7.85^2 \times 0.0400 = 2.47$ m s^{-2}; the magnitude is maximum at $x = \pm A$.

Teacher prompt / common error: Maximum speed is at equilibrium. Maximum acceleration magnitude is at the turning positions, where velocity is zero.

5. At one instant the piston has displacement $x = -24.0$ mm and is moving in the negative direction. Use $v = \pm\omega\sqrt{(A^2 - x^2)}$ to calculate its signed velocity. Calculate its acceleration at the same instant and state how the acceleration direction compares with the velocity direction. [4 marks]

Answer / mark points: $|v| = 7.85\sqrt{(0.0400^2 - 0.0240^2)} = 7.85 \times 0.0320 = 0.251$ m s^{-1}. The piston is moving in the negative direction, so $v = -0.251$ m s^{-1}. $a = -\omega^2 x = -(7.85^2)(-0.0240) = +1.48$ m s^{-2}. The acceleration is positive, opposite to the velocity and towards equilibrium.

Teacher prompt / common error: The square-root equation gives a magnitude until the sign is chosen from the direction. Acceleration at negative displacement is positive and towards equilibrium.

6. Starting with $x = A\cos(\omega t)$, derive $v = \pm\omega\sqrt{(A^2 - x^2)}$. Your derivation must show why there are two possible signs and how $v_{max} = \omega A$ follows. [3 marks]

Answer / mark points: Differentiate $x = A\cos(\omega t)$: $v = -A\omega\sin(\omega t)$. Squaring and using $\sin^2(\omega t) = 1 - \cos^2(\omega t)$ and $\cos(\omega t) = x/A$ gives $v^2 = \omega^2(A^2 - x^2)$. Taking the square root gives $v = \pm\omega\sqrt{(A^2 - x^2)}$; the sign depends on the direction of motion. At $x = 0$ the magnitude is greatest, so $v_{max} = \omega A$.

Teacher prompt / common error: The ± sign appears when taking the square root; it represents the two possible directions at the same displacement.

3. Amplitude and energy calculations

7. The calibration is repeated with amplitude 60.0 mm but with the same piston-spring assembly. State the new period, calculate the new maximum speed, and determine the factor by which the maximum kinetic energy changes. [3 marks]

Answer / mark points: For an ideal simple harmonic oscillator the period is independent of amplitude, so T remains 0.800 s. $v_{max} = \omega A = 7.85 \times 0.0600 = 0.471$ m s^{-1}. Since maximum kinetic energy is proportional to v_{max}^2 and hence A^2, the factor is $(0.0600/0.0400)^2 = 2.25$.

Teacher prompt / common error: Isochronous means period is independent of amplitude for ideal SHM. Maximum energy changes as A^2, not A.

8. The effective oscillating mass is 1.60 kg. For the 40.0 mm amplitude calibration, calculate the total mechanical energy. At $x = 24.0$ mm, calculate the elastic potential energy and kinetic energy, then use the kinetic energy to calculate the piston speed. [5 marks]

Answer / mark points: $E_{total} = 1/2\, m\omega^2 A^2 = 1/2 \times 1.60 \times 7.85^2 \times 0.0400^2 = 7.90 \times 10^{-2}$ J. At $x = 0.0240$ m, $E_{PE} = 1/2\, m\omega^2 x^2 = 2.84 \times 10^{-2}$ J. $E_{KE} = E_{total} - E_{PE} = 5.05 \times 10^{-2}$ J. Using $E_{KE} = 1/2\, mv^2$ gives $v = \sqrt{(2E_{KE}/m)} = 0.251$ m s^{-1}, consistent with Question 5.

Teacher prompt / common error: Keep all displacements in metres. At a stated x, kinetic plus elastic potential energy must equal the constant total.

4. Energy-displacement graphs

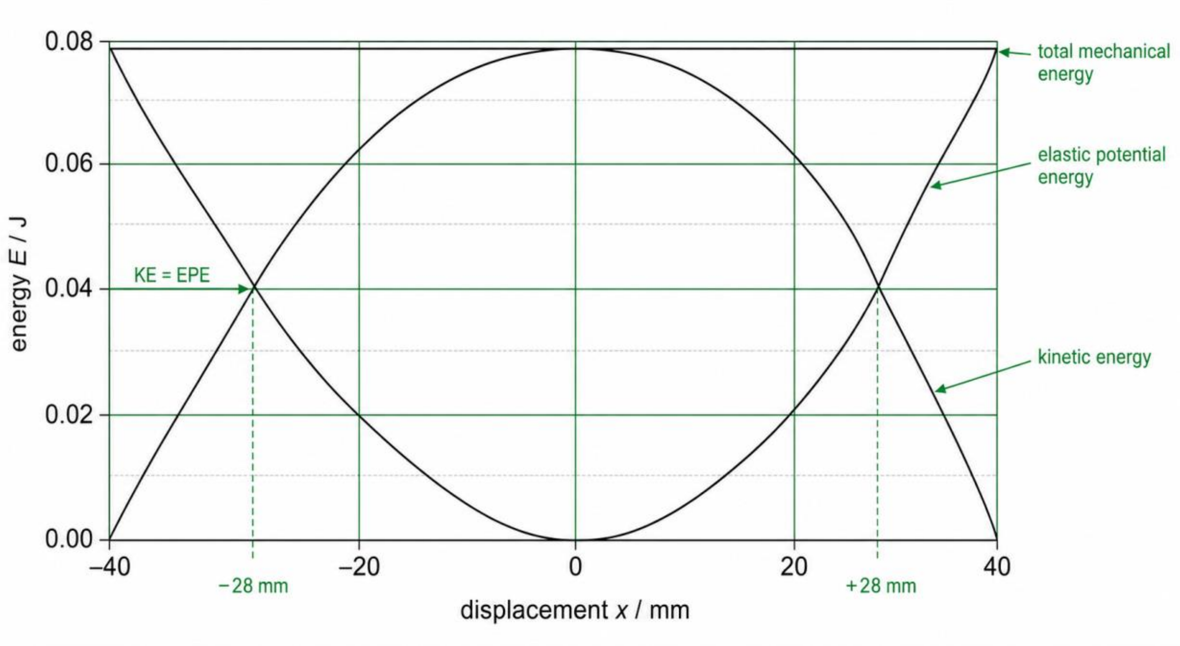

Figure 3 solution: energy curves and the equal-energy positions

9. Figure 3 shows three energy-displacement curves for the free calibration run. (a) Label the total mechanical energy, elastic potential energy and kinetic energy curves. (b) Estimate the magnitude of the displacement where the kinetic and elastic potential energies are equal. (c) Describe the energy transfers as the piston moves from the negative extreme, through equilibrium, to the positive extreme. [5 marks]

Answer / mark points: (a) The horizontal line is total mechanical energy. The upward-opening parabola is elastic potential energy. The downward-opening parabola is kinetic energy. (b) The curves cross at |x| ≈ 28 mm, close to A/√2. (c) From x = -A to x = 0, elastic potential energy decreases while kinetic energy increases; kinetic energy is maximum and elastic potential energy minimum at equilibrium. From x = 0 to x = +A the transfer reverses. The total remains constant in this ideal free calibration.

Teacher prompt / common error: Energy depends on x^2, so the graph is symmetric about x = 0. Equal energy occurs at |x| = A/√2.

5. Displacement, velocity and acceleration

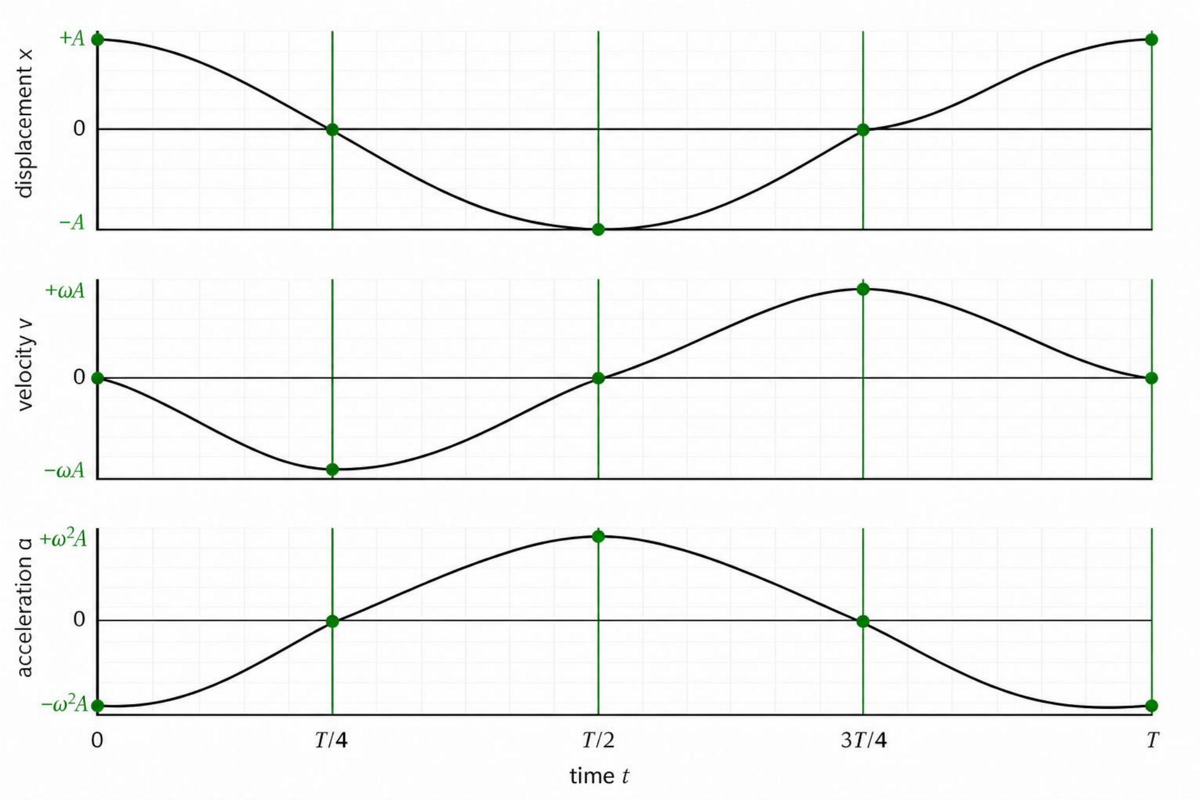

Figure 4 solution: aligned checkpoints and phases over one complete cycle

10. On Figure 4, draw displacement x, velocity v and acceleration a against time for one complete cycle. The piston begins at x = +A at t = 0. Show the correct zero crossings and label the positive and negative extrema of all three quantities. [6 marks]

Answer / mark points: Displacement is a cosine curve: +A, 0, -A, 0, +A at 0, T/4, T/2, 3T/4, T. Velocity is its gradient: 0, -ωA, 0, +ωA, 0 at the same times. Acceleration is $-\omega^2x$: $-\omega^2A$, 0, $+\omega^2A$, 0, $-\omega^2A$. The three graphs must share the same quarter-period guide positions.

Teacher prompt / common error: Use differentiation language: v is the gradient of x-t and a is the gradient of v-t. Acceleration is exactly opposite in phase to displacement.

6. Reading the phase relationships

11. Use your graphs in Figure 4. (a) State x, v and a at t = T/4. (b) State the time offset between corresponding extrema of displacement and velocity. (c) State the phase relationship between displacement and acceleration. **[3 marks]**

Answer / mark points: (a) At $t = T/4$: $x = 0$, $v = -\omega A$ and $a = 0$. (b) Corresponding extrema of displacement and velocity are separated by $T/4$. (c) Acceleration is in antiphase with displacement: phase difference π rad or 180°.

Teacher prompt / common error: At T/4 the piston crosses equilibrium moving in the negative direction. A quarter-period time shift corresponds to $\pi/2$ rad.

12. During water pumping the crank mechanism enforces an approximately sinusoidal piston displacement. A student concludes that the piston's total mechanical energy must therefore be constant. Explain why this conclusion does not follow and identify the important energy transfers during pumping. **[3 marks]**

Answer / mark points: A sinusoidal displacement describes the motion but does not show that the piston is isolated or undamped. The motor continuously transfers energy to the driven piston. The piston transfers energy to the water by doing pressure work and energy is also dissipated thermally by fluid resistance, friction and turbulence. Therefore the piston's instantaneous mechanical energy need not be constant even when its displacement is sinusoidal.

Teacher prompt / common error: Sinusoidal motion does not prove energy conservation. Separate the ideal free calibration model from the externally driven, dissipative pumping mode.

Language and model check

Say ‘velocity is negative’ or ‘the piston moves left’; acceleration has a magnitude and direction and is always towards equilibrium in SHM.

A sinusoidal x-t graph does not by itself prove that mechanical energy is constant. Energy conservation requires the chosen system and transfers to be identified.