OCR A A Level Simple Harmonic Motion 3 solutions

Matching solutions and teaching prompts for the class review

Specification points covered

- 5.3.1(c)(i) Define simple harmonic motion using $a = -\omega^2 x$.
- 5.3.1(e) Derive velocity for SHM as $v = \pm\omega\sqrt{(A^2 - x^2)}$ and identify $v_max = \omega A$.
- 5.3.2(a) Describe interchange between kinetic and potential energy during SHM.
- 5.3.2(b) Interpret energy-displacement graphs for a simple harmonic oscillator.
- 5.3.3(a) Distinguish between free and forced oscillations.
- 5.3.3(b)(i) Describe the effects of damping on an oscillating system.
- 5.3.3(b)(ii) Observe and describe forced and damped oscillations for a range of systems.
- 5.3.3(c) Define resonance and natural frequency and describe resonance conditions.
- 5.3.3(d) Sketch and interpret amplitude-frequency response graphs for forced oscillations.
- 5.3.3(e) Give practical examples of forced oscillations and resonance in real systems.

Teaching focus

Keep absolute elastic energy separate from the displacement-dependent energy used in the SHM model. The pre-tensioned springs store energy even at equilibrium.
Use exact quantity language: amplitude decreases under damping; the oscillator is driven at a frequency; energy is transferred thermally; resonance produces a large amplitude.

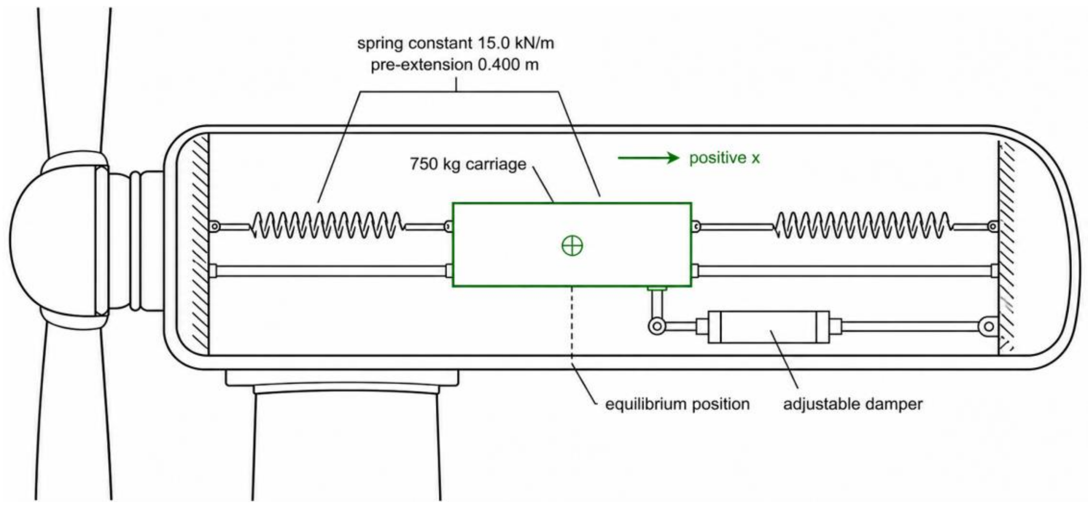

Figure 1: carriage, two pre-tensioned springs and damper

1. Natural motion of the carriage

1. The carriage is displaced a distance x to the right. The left spring extension becomes $e_0 + x$ and the right spring extension becomes $e_0 - x$. Use the two spring forces to show that the resultant horizontal force is $F = -2k_s x$. Explain how this establishes SHM and state the effective force constant of the spring combination. **[4 marks]**

Answer / mark points: Take right as positive. The left spring pulls left with force $-k_s(e_0 + x)$, while the right spring pulls right with force $+k_s(e_0 - x)$. Therefore $F = k_s(e_0 - x) - k_s(e_0 + x) = -2k_s x$. The resultant force is directly proportional to displacement and opposite in direction, so it is a restoring force and the motion satisfies the SHM condition. The effective force constant is $k_e$ff = $2k_s$ = 30.0 kN m^{-1}.
Teacher prompt / common error: The pre-extension terms cancel in the resultant force but not in the stored elastic energy. Require the negative sign and the two-part SHM definition.

2. The carriage mass is 750 kg and each spring has force constant k_s = 15.0 kN m^{-1}. Calculate the natural angular frequency ω_0, natural frequency f_0 and natural period T_0 of the carriage. **[3 marks]**

Answer / mark points: $\omega_0 = \sqrt{(k_e\text{ff}/m)} = \sqrt{(3.00 \times 10^4/750)} = \sqrt{40.0} = 6.32$ rad s^{-1}. $f_0 = \omega_0/(2\pi) = 6.32/(2\pi) = 1.01$ Hz. $T_0 = 1/f_0 = 0.993$ s.
Teacher prompt / common error: Use the effective combination constant $2k_s$, not a single spring constant. Distinguish angular frequency from frequency.

3. In a low-damping release test the carriage amplitude is 0.120 m. Calculate the maximum speed and maximum acceleration magnitude of the carriage. **[2 marks]**

Answer / mark points: $v_m$ax = $\omega_0 A$ = 6.32 × 0.120 = 0.759 m s^{-1}. $a_m$ax = $\omega_0^2 A$ = 40.0 × 0.120 = 4.80 m s^{-2}.
Teacher prompt / common error: Maximum speed is at equilibrium; maximum acceleration magnitude is at the turning positions. Keep amplitudes in metres.

2. Springs that are always in tension

4. At equilibrium, each spring is already extended by $e_0 = 0.400$ m. (a) Starting with $U = 1/2k_s(e_0 + x)^2 + 1/2k_s(e_0 - x)^2$, show that the total elastic potential energy is $U = k_se_0^2 + k_sx^2$. (b) Calculate the minimum value of U and its value at x = +0.120 m. (c) Calculate both spring extensions at x = +0.120 m and use them to explain why neither spring becomes slack and why the elastic potential energy is never zero. (d) Explain why an SHM energy graph may nevertheless show potential energy equal to zero at equilibrium. [7 marks]

Answer / mark points: (a) Expanding gives $U = 1/2k_s(e_0^2 + 2e_0x + x^2) + 1/2k_s(e_0^2 - 2e_0x + x^2)$. The cross-terms cancel, so $U = k_se_0^2 + k_sx^2$. (b) $U_{m}in = k_se_0^2 = 1.50 \times 10^4 \times 0.400^2 = 2.40 \times 10^3$ J. At x = 0.120 m, $U = 2400 + 1.50 \times 10^4 \times 0.120^2 = 2616$ J. (c) The extensions are $e_0 + x = 0.520$ m and $e_0 - x = 0.280$ m. Both are positive, so both springs remain in tension and each stores elastic energy; the total U cannot be zero. (d) An SHM energy graph normally chooses the minimum as the reference zero. It plots $U_{o}sc = U - U_{m}in = k_sx^2 = 1/2k_{e}ffx^2$, not the absolute elastic energy stored in the pre-tensioned springs.

Teacher prompt / common error: The absolute spring energy has a non-zero minimum. Choosing a zero of potential energy is a reference convention, not a claim that the springs are unstretched.

3. Damped free oscillation

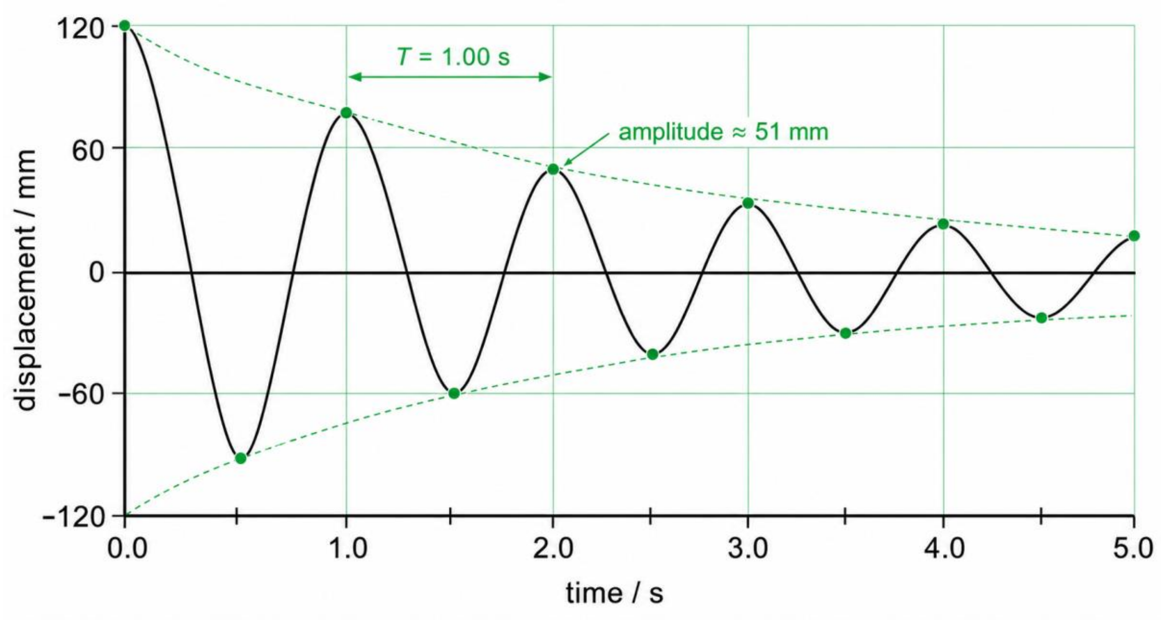

Figure 2 solution: period, amplitude and decaying envelopes

5. Figure 2 is a displacement-time record after the carriage is released from rest. (a) Determine the period. (b) Read the amplitude at t = 2.0 s and calculate the fraction of the initial oscillation energy that remains, using E proportional to A^2. (c) At the first equilibrium crossing, state the directions of the carriage velocity and the damping force. **[4 marks]**

Answer / mark points: (a) Successive positive maxima are separated by 1.00 s, so T = 1.00 s. (b) At 2.0 s, A is about 51 mm. Since oscillation energy is proportional to A^2, $E_2/E_0 = (51/120)^2 = 0.181$, so about 18% remains. (c) At the first equilibrium crossing the carriage is moving left, so its velocity is negative. The damping force acts opposite to the velocity, so it is directed right and is positive.

Teacher prompt / common error: Read period between same-phase points. Energy falls as amplitude squared. A damping force opposes velocity, not necessarily acceleration.

6. Describe the energy transfers from the initial positive turning position, through the first equilibrium crossing, to the first negative turning position. Your answer must distinguish kinetic energy, the displacement-dependent elastic potential energy and energy dissipated by the damper. **[4 marks]**

Answer / mark points: At the initial positive turning position the carriage velocity and kinetic energy are zero, while the displacement-dependent elastic potential energy is maximum. As the carriage moves towards equilibrium, this potential energy decreases and transfers mainly to kinetic energy, while the damper transfers some energy thermally to its fluid and the surroundings. At equilibrium the kinetic energy is maximum and the displacement-dependent elastic potential energy is zero relative to its chosen minimum, although the two pre-tensioned springs still store 2400 J. From equilibrium to the negative turning position, kinetic energy transfers back to displacement-dependent elastic potential energy and more is dissipated thermally. The total oscillation energy therefore decreases.

Teacher prompt / common error: Use transfer language. The displacement-dependent potential can be zero at equilibrium while the actual springs still store their pre-tension energy.

4. Free, forced and resonant motion

7. Engineers use two tests. In Test 1 the carriage is displaced and released while the nacelle is stationary. In Test 2 a rotating imbalance produces a continuing periodic force on the nacelle. State which test is a free oscillation and which is a forced oscillation, and give the reason for each choice. [2 marks]

Answer / mark points: Test 1 is a free oscillation: after the initial displacement and release there is no continuing periodic driving force, so the system oscillates at approximately its natural frequency while damping reduces the amplitude. Test 2 is a forced oscillation because the rotating imbalance supplies a continuing periodic force; the steady-state oscillation occurs at the driving frequency.

Teacher prompt / common error: Free does not mean undamped. Forced means a continuing periodic driver; the steady-state frequency is the driving frequency.

8. Define natural frequency and resonance. Explain why a forced oscillator has a large steady-state amplitude when the driving frequency is close to its natural frequency. [3 marks]

Answer / mark points: The natural frequency is the frequency at which a system oscillates freely after being displaced and released. Resonance is the large-amplitude response of a forced oscillator when the driving frequency is equal or close to the natural frequency. Under this condition the driver transfers energy to the oscillator efficiently on successive cycles, so the steady-state amplitude becomes large until energy input per cycle equals energy dissipated per cycle.

Teacher prompt / common error: Resonance requires the driving and natural frequencies to be equal or close. The large amplitude follows from efficient energy transfer over repeated cycles.

5. Interpreting the resonance curves

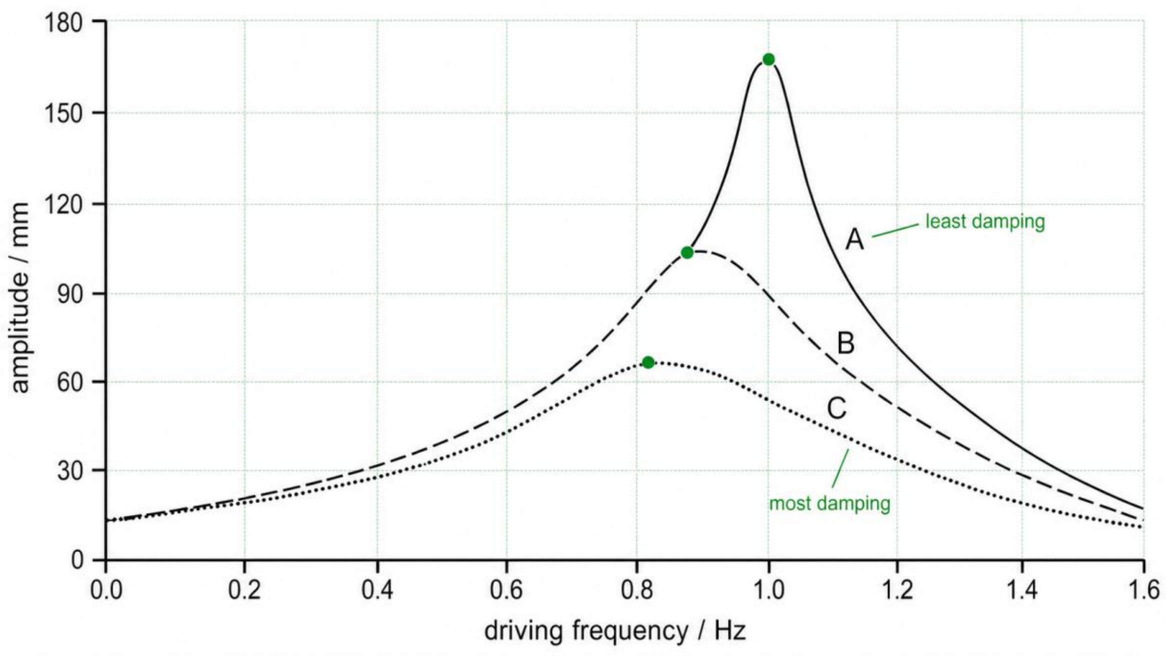

Figure 3 solution: least- and most-damped response curves

9. Figure 3 shows the amplitude-frequency responses for damper settings A, B and C. (a) Determine the resonant frequency for setting A and identify the most heavily damped setting. (b) Describe how increasing the damping changes the height, width and position of the resonance peak. [5 marks]

Answer / mark points: (a) Setting A has its maximum amplitude at about 1.00 Hz, so its resonant frequency is about 1.00 Hz. Setting C has the lowest, broadest peak and is the most heavily damped. (b) Increasing damping reduces the maximum amplitude and makes the response peak lower and broader. The frequency at the maximum shifts slightly to a lower value.

Teacher prompt / common error: More damping gives a lower, broader peak; OCR accepts a small shift of the peak to lower frequency. Do not say the frequency itself becomes damped.

10. At one operating condition the rotor produces one force pulse per revolution and rotates at 58.0 rev min^{-1}. Calculate the driving frequency and explain why this operating condition requires careful damping design. [2 marks]

Answer / mark points: $f_d$rive = 58.0/60 = 0.967 Hz. This is close to the calculated natural frequency of 1.01 Hz, so the carriage is driven close to resonance and may reach a large amplitude unless sufficient damping is used.

Teacher prompt / common error: Convert revolutions per minute to hertz before comparing with the natural frequency.

6. Applying the design constraints

Table 1: measured performance at 58.0 rev min^{-1}

Damper setting	Carriage amplitude / mm	Nacelle acceleration / m s^{-2}
A	160	0.014
B	105	0.019
C	62	0.030

11. Table 1 gives measured performance at 58.0 rev min^{-1}. The design requires carriage amplitude below 120 mm and nacelle acceleration below 0.025 m s^{-2}. Select the only acceptable damper setting and justify your decision using both requirements. [3 marks]

Answer / mark points: Setting A fails the amplitude requirement because 160 mm is greater than 120 mm. Setting C passes the amplitude requirement but fails the acceleration requirement because 0.030 m s^{-2} is greater than 0.025 m s^{-2}. Setting B is the only acceptable choice: 105 mm is below 120 mm and 0.019 m s^{-2} is below 0.025 m s^{-2}.

Teacher prompt / common error: A valid engineering choice must satisfy every stated constraint. Pupils should explicitly reject A and C rather than merely name B.

12. With the selected setting, the damper dissipates 210 J during each steady oscillation cycle at 58.0 rev min^{-1}. Calculate the mean power dissipated. Explain why the amplitude can remain constant even though energy is continuously dissipated. [3 marks]

Answer / mark points: The oscillation frequency is 58.0/60 = 0.967 s^{-1}. Mean power = energy per cycle × cycles per second = 210 × 0.967 = 203 W. In the steady state the periodic driver supplies 210 J per cycle, equal to the energy dissipated by the damper, so the oscillator energy and amplitude remain constant from cycle to cycle.

Teacher prompt / common error: In a steady forced oscillation, amplitude is constant because energy input per cycle equals energy dissipated per cycle, not because no energy is lost.

7. Measuring a response curve

13. Describe a method the engineers could use to obtain an amplitude-frequency response curve and compare two damper settings. Include how the oscillator is driven and measured, the variables that must be controlled, how the frequency range should be sampled, and how the results should be processed. [5 marks]

Answer / mark points: Attach a displacement sensor or calibrated video marker to the carriage and use a tachometer or the motor controller to measure the driving frequency. Drive the nacelle using the same rotating imbalance throughout, keeping the imbalance mass, radius or drive-force amplitude, carriage mass and springs unchanged. Choose a range of driving frequencies on both sides of the expected 1.0 Hz resonance. At each frequency allow transient free oscillations to decay so that a steady state is reached, then record several cycles and determine the amplitude from half the peak-to-peak displacement. Repeat and calculate a mean. Use smaller frequency intervals near the maximum, where the response changes rapidly, and wider intervals far from the peak. Plot mean amplitude against measured driving frequency with sensible scales and uncertainty bars if available. Repeat the same frequency sequence for the second damper setting, keeping the controlled variables unchanged, and compare peak amplitude, peak frequency and peak width. Stop any run before the carriage reaches its safe travel limit.

Teacher prompt / common error: Full credit needs a measured drive frequency, constant drive amplitude, steady-state readings, repeats, closer frequency spacing near resonance and a graph for each damping setting.

Language and model check

Say ‘the amplitude decreases’ during damped motion. Do not say that the velocity or frequency is damped.

At steady forced amplitude, energy is still transferred: energy supplied by the driver per cycle equals energy dissipated per cycle.

A zero on a potential-energy graph is a chosen reference. It does not prove that a stretched spring contains no elastic energy.