Radians sine cosine and phase solutions

Matching worked solutions and teaching prompts | 38 marks

AQA A Level 7408 and OCR A A Level H556 | Mathematical skills

Skills practised and specification links

Use radians and angular frequency; choose sine/cosine from initial conditions; draw periodic traces; interpret phase, inverse-trig branches and squared functions.
AQA maths MS 0.6, 2.2–2.3, 3.1–3.2, 3.12, 4.5, 4.7; selected applications 3.6.1.1–3.6.1.3.
OCR A maths M0.6, M2.2–M2.3, M3.1–M3.2, M3.12, M4.5, M4.7; 5.2.1(a)–(c); 5.3.1(a), (b), (d), (g); 5.3.2(a).

An inspection scanner moves a camera head across a surface. Each stage is a separate ideal test. Ignore damping; use a ruler and calculator. Check calculator mode and show working. Times in equations are in seconds.

1 Connect radians to the timing disc [4 marks]

Radian angle θ = s/r (arc/radius, same length units). One turn = 2π rad = 360°.

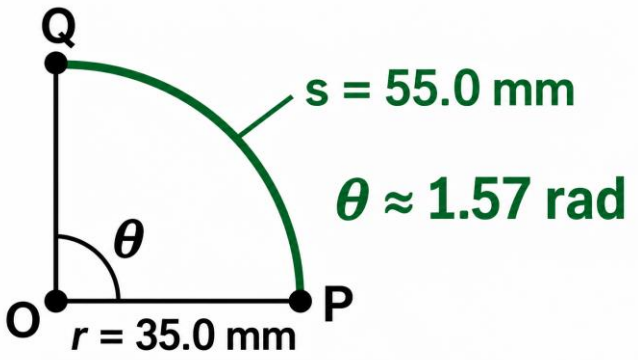


Figure 1 A marked arc on the scanner's circular timing disc; schematic.

(a) Express one quarter of a complete turn in radians. [1 mark]
(b) Convert 135° to radians, giving an exact multiple of π. [1 mark]
(c) Use the values in Figure 1 to calculate θ. Decide whether it is approximately a quarter turn. [2 marks]

Worked answers and marking

(a) ¼ × 2π = π/2 rad ≈ 1.57 rad. [1]
(b) 135 × π/180 = 3π/4 rad. [1]
(c) θ = 55.0/35.0 = 1.5714… rad ≈ 1.57 rad. [1] This agrees with π/2 to the precision of the data, so it is approximately a quarter turn (about 90.0°). [1] Equal length units cancel; do not convert one length only.

Teaching prompt A radian is an arc-length/radius ratio. The radian label identifies the angle convention even though the ratio itself is dimensionless.

2 Set the drive speed and calculator mode [6 marks]

The timing disc turns uniformly at f = 2.00 revolutions per second. One complete disc turn sets one complete scanner cycle. Let ω be the angular speed of the disc and the angular frequency of the scanner. Use $\omega = 2\pi f = 2\pi/T$.

(a) Calculate the period T of one scanner cycle. [1 mark]

(b) Calculate ω and explain why it is not numerically equal to f. [2 marks]

(c) Calculate the disc's angular advance in 0.0750 s, in radians. [1 mark]

(d) A pupil enters cos(ωt) at t = 0.0750 s in degree mode and obtains almost 1. Explain the error and give the correct value of cos(ωt). [2 marks]

Worked answers and marking

(a) $T = 1/f = 1/2.00 = 0.500$ s. [1]

(b) $\omega = 2\pi \times 2.00 = 4\pi \approx 12.6$ rad s^{-1}. [1] Frequency counts cycles per second; each cycle advances the phase by 2π radians, so angular frequency is 2π times f. [1]

(c) $\Delta\theta = \omega\Delta t = 4\pi \times 0.0750 = 0.300\pi = 0.942$ rad. [1] No extra factor of 2π is needed after ω has been calculated.

(d) ωt = 0.942… is a radian angle, so the calculator must use radian mode. [1] $\cos(0.300\pi) = 0.588$. [1] Degree mode instead interprets 0.942 as degrees, giving about 0.9999.

Teaching prompt Use 2π once to convert cycles into radians. Match the calculator mode to the angle inside the function, not to the length or time unit.

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3 Choose an equation from the starting motion [8 marks]

Two separate test heads have amplitude A = 6.00 mm and angular frequency $\omega = 4\pi$ rad s^{-1}. Displacement is measured from equilibrium, positive to the right. Head P starts at its positive extreme at t = 0 and then moves left. Head Q starts at equilibrium at t = 0, moving left. Keep A positive.

Test the starting value AND the direction just after t = 0. For example, cos 0 = 1 but sin 0 = 0; for a small positive radian angle, sine is positive. A correct starting displacement alone does not establish the correct equation.

(a) Write a displacement equation for head P, with x_P in metres and t in seconds. [2 marks]
(b) Write a displacement equation for head Q, with x_Q in metres and t in seconds. [2 marks]
(c) A technician proposes $x_Q = A\sin(\omega t)$. Explain why testing only t = 0 fails to reveal its error. [1 mark]
(d) Use the correct equation for Q to calculate its displacement at t = 0.0750 s. [1 mark]
(e) Find the first positive time when P reaches $x_P = -3.00$ mm. Identify the quarter-cycle interval containing that time. [2 marks]

Worked answers and marking

(a) $x_P = 6.00 \times 10^{-3}\cos(4\pi t)$, with x_P in m. Correct positive-cosine form and angular frequency. [1] Correct conversion of amplitude to metres. [1] At t = 0 it gives +A; immediately afterwards displacement decreases.

(b) $x_Q = -6.00 \times 10^{-3}\sin(4\pi t)$, with x_Q in m. Negative-sine form gives the correct initial direction. [1] Correct SI amplitude and angular frequency. [1] Accept an equivalent phase-shifted cosine expression.

(c) Both expressions give x = 0 at t = 0, but +A sin(ωt) initially increases, so it predicts motion to the right instead of left. [1]

(d) $x_Q = -6.00 \times 10^{-3}\sin(0.300\pi) = -4.85 \times 10^{-3}$ m, or −4.85 mm. [1] The negative sign means left of equilibrium, not a negative amplitude.

(e) $\cos(4\pi t) = -3.00/6.00 = -0.500$; the first positive phase is $2\pi/3$. [1] $t = (2\pi/3)/(4\pi) = 1/6$ s = 0.167 s, between T/4 = 0.125 s and T/2 = 0.250 s. [1] A later valid cosine branch does not give the first time.

Teaching prompt Amplitude is positive. The sign or phase in the equation encodes the starting state. Check a small positive time to distinguish opposite equilibrium crossings.

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4 Read one trace and draw another [7 marks]

Figure 2 is a recording for head 1 in a new test. Its motion is sinusoidal. In part (d), add a trace for head S, which has the same amplitude and period but crosses equilibrium moving in the positive direction at t = 0.

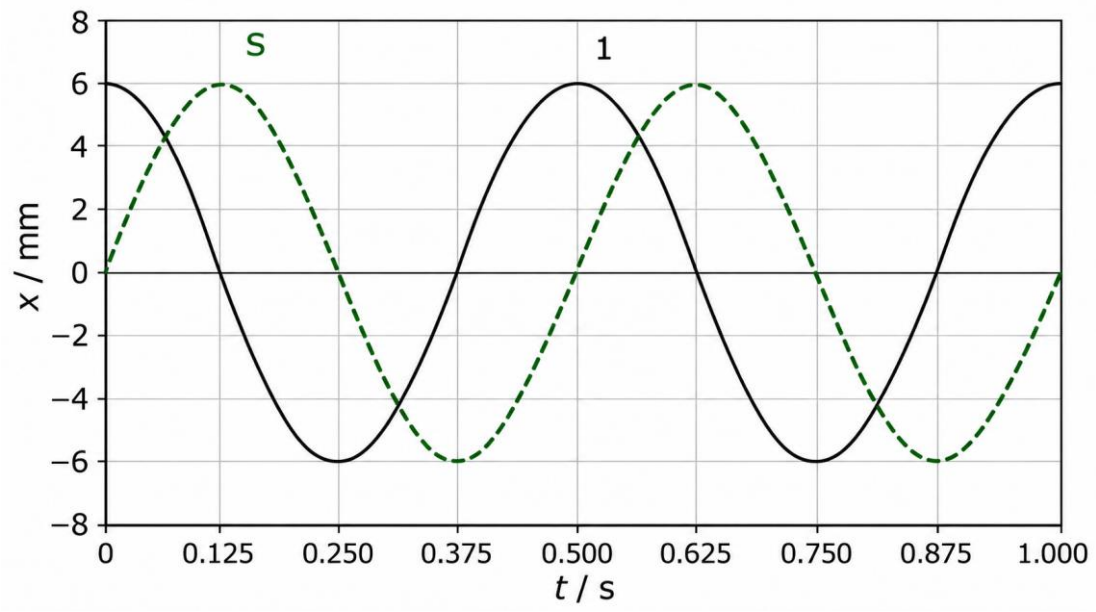


Figure 2 Black: head 1. Dashed green: head S, initially moving in the positive direction.

t / s	0	0.125	0.250	0.375	0.500
x_1 / mm					

(a) Read the amplitude and period of head 1 from the graph. [2 marks]

(b) Write an equation for head 1, with x_1 in millimetres and t in seconds. [1 mark]

(c) Complete the table with the first cycle's quarter-cycle displacements for head 1. [1 mark]

(d) Add head S to Figure 2 over the full 0–1.00 s interval. Use a dashed curve and label it S. [2 marks]

(e) Explain why this displacement–time graph is not a picture of the scanner's shape at one instant. [1 mark]

Worked answers and marking

(a) Amplitude = 6.0 mm. [1] Period = 0.500 s. [1] Accept about 5.8–6.2 mm and 0.48–0.52 s from a sensible reading. Peak-to-peak displacement is 12 mm, not the amplitude.

(b) $x_1 = 6.0 \cos(4\pi t)$, with x_1 in mm. [1] Accept consistent graphical values for A and T, using $\omega = 2\pi/T$ and the positive-maximum start.

(c) In time order: +6.0, 0, −6.0, 0, +6.0 mm. All correct. [1] Each entry is a displacement at one time.

(d) S passes through (0,0), (0.125,+6), (0.250,0), (0.375,−6), (0.500,0), then repeats. Correct quarter-cycle positions and initial positive direction. [1] A smooth sinusoidal dashed curve with the same amplitude/period through two full cycles, labelled S. [1] It corresponds to $x_S = 6.0 \sin(4\pi t)$, in mm.

(e) The horizontal axis is time: the trace shows one head's displacement at successive times, not different points' positions at one instant. [1]

Teaching prompt Read the units before writing the equation. A time trace follows one moving object; a spatial snapshot uses position along an object as its horizontal variable.

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5 Synchronise the two scanner heads [8 marks]

Use head 1 and head S from Figure 2. The controller records each positive maximum of S 0.125 s after the corresponding maximum of head 1. Their common period is 0.500 s. For calculations in this section use $x_1 = 6.00\cos(4\pi t)$ and $x_S = 6.00\sin(4\pi t)$, with displacements in mm.

Phase difference = 2π × time delay / period. State which signal leads or lags. An inverse-trig key returns one principal angle; check the other branches and restrict answers to the requested time interval.

(a) Calculate the phase difference in radians and in degrees. [2 marks]

(b) State which head leads and justify the direction of the lead. [1 mark]

(c) Find all times in $0 \le t < 0.500$ s when $x_S = +3.00$ mm. Show both phase solutions. [3 marks]

(d) A trigger is enabled only for $0.180 \le t \le 0.250$ s and requires $x_S = +3.00$ mm while S moves in the negative direction. Which event triggers it? Explain. [2 marks]

Worked answers and marking

(a) $\Delta\varphi = 2\pi(0.125/0.500) = \pi/2$ rad ≈ 1.57 rad. [1] This is 90°. [1]

(b) Head 1 leads S by π/2 rad, because its corresponding positive maximum occurs earlier. Equivalently, S lags head 1. [1] A phase magnitude alone does not specify lead/lag.

(c) $\sin(4\pi t) = 0.500$; principal phase = π/6. [1] The second phase in this cycle is 5π/6. [1] $t = (\pi/6)/(4\pi) = 1/24$ s = 0.0417 s and $t = (5\pi/6)/(4\pi) = 5/24$ s = 0.208 s. Both times. [1] Adding 2π would move to a later cycle.

(d) The event at t = 0.208 s lies inside the enabled interval. [1] It is after S's positive maximum at 0.125 s and before its zero crossing at 0.250 s, so displacement is decreasing and S moves in the negative direction. [1] The principal-branch event at 0.0417 s is outside the window.

Teaching prompt Name the reference head and the direction of phase difference. One displacement can occur twice per cycle with opposite directions of motion; use both the time window and direction.

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6 Interpret a squared sine in an energy model [5 marks]

In a separate free-oscillation calibration, the spring-mounted head starts at positive maximum displacement and oscillates without energy loss. Its displacement is $x = A\cos(\omega t)$, with $\omega = 4\pi$ rad s^{-1}. The total mechanical energy E is constant. Use the supplied relation $K/E = \sin^2(\omega t)$ for its kinetic-energy fraction. Here $\theta = \omega t$.

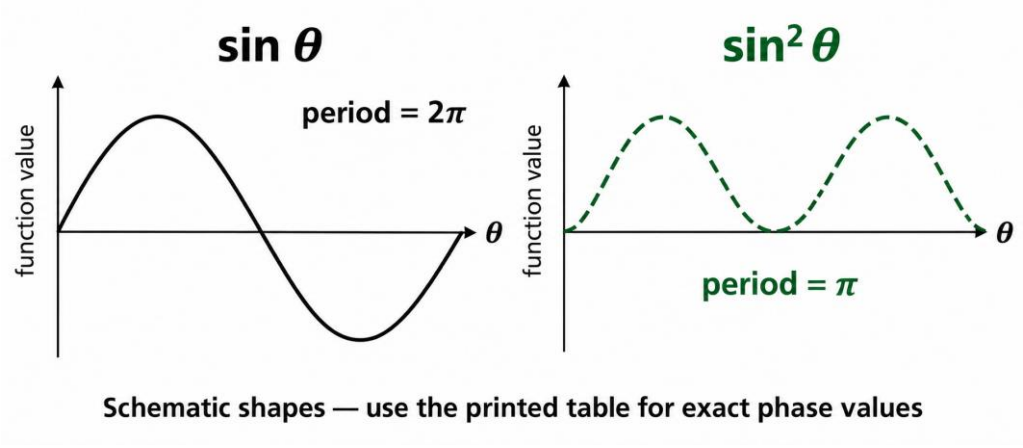


Figure 3 Qualitative function shapes. Use the table and equations for exact phase values.

θ / rad	0	π/2	π	3π/2	2π
sin θ	0	+1	0	−1	0
$\sin^2$ θ	0	1	0	1	0

(a) Calculate K/E when θ = 0.800 rad. [1 mark]

(b) Find the repeat time of K/E. Explain why it is half the displacement period. [2 marks]

(c) At θ = 2.00 rad, a pupil enters $\sin(\theta^2)$ and obtains −0.757. Give the correct bracketed calculation and fraction, then explain why the negative value is impossible for K/E. [2 marks]

Worked answers and marking

(a) $K/E = [\sin(0.800)]^2 = 0.5146\ldots \approx 0.515$. [1] Radian mode; the result is a dimensionless fraction.

(b) $\sin^2\theta$ repeats after a phase advance of π, so its repeat time is $\pi/\omega = \pi/(4\pi) = 0.250$ s. [1] Sine changes sign after π, but squaring removes the sign; the same energy fraction therefore repeats twice per displacement cycle. [1] The phase period π rad is not a time of π seconds.

(c) The required calculation is $[\sin(2.00)]^2 = 0.8268\ldots \approx 0.827$, not $\sin(2.00^2)$. [1] K and E are non-negative here with $0 \le K \le E$, so $0 \le K/E \le 1$; a negative kinetic-energy fraction is impossible. [1]

Teaching prompt $\sin^2\theta$ means $[\sin\theta]^2$. Square the function value, not the angle. Keep phase periods in radians separate from repeat times in seconds.

Final check Check angle units, initial value and direction, phase branches, the time window and calculator brackets.

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