Proportionality ratios and scaling solutions

Matching worked solutions and teaching prompts | 43 marks

AQA A Level 7408 and OCR A A Level H556 | Mathematical skills

Skills practised

Use factors, powers and percentages; compare models by ratios; identify what must remain fixed.
AQA maths MS 0.1, 0.3, 0.5, 2.1–2.3, 4.3; selected applications 3.4.1.7–3.4.1.8; 3.4.2.1; 3.5.1.3–3.5.1.4; 3.7.2.2.
OCR A maths M0.1, M0.3, M0.5, M2.1–M2.3, M4.3; selected applications 3.2.4(a); 3.3.2(a); 3.3.3(a); 4.2.4(a)(i), (b); 4.2.5(a), (b); 5.4.1(b); 5.4.2(b).

A remote environmental station is being redesigned. Compare conductors, power losses, battery life and detector positions; then transfer the method to a planetary probe. Models are supplied. Use a calculator; give ratios without units and physical results with units. Keep unrounded factors until the final step.

1 Turn a change into a factor [6 marks]

First divide the new value by the original value. If $y = Cx^n$ and C is unchanged, $y_2/y_1 = (x_2/x_1)^n$. For several changing variables, multiply their factors with the correct powers.
Percentage change = (factor − 1) × 100%. A factor of 1.30 means 130% of the original, an increase of 30%.
Worked example: $E_k = \frac{1}{2}mv^2$. If speed stays fixed and mass increases by 20%, $E_{k2}/E_{k1} = 1.20$. The constant ½ cancels, and the speed factor is 1.

(a) For $y \propto x^2$, x becomes 1.50 times its original value. State the factor by which y changes. [1 mark]

(b) For $y \propto 1/x$, x becomes 0.80 times its original value. State the factor by which y changes. [1 mark]

(c) The station's inspection trolley is redesigned with 0.80 times its original mass and 1.50 times its original speed. Find the kinetic-energy factor and percentage increase. Use $E_k = \frac{1}{2}mv^2$. [2 marks]

(d) A sensor has the same usable battery energy after an upgrade, but its constant power draw increases by 25%. Find its operating-time factor and percentage decrease. Use E = Pt. [2 marks]

Worked answers and marking

(a) $y_2/y_1 = 1.50^2 = 2.25$. [1] This is 225% of the original, not a 225% increase.

(b) $y_2/y_1 = 1/0.80 = 1.25$. [1] An inverse relationship reverses the direction of the change.

(c) $E_{k2}/E_{k1} = (m_2/m_1)(v_2/v_1)^2 = 0.80 \times 1.50^2 = 1.80$. [1] Percentage increase = (1.80 − 1) × 100% = 80%. [1]

(d) At fixed usable energy, $t_2/t_1 = P_1/P_2 = 1/1.25 = 0.80$. [1] The operating time decreases by 20%, not 25%. [1]

Teaching prompt Say what the factor multiplies. Keep "percentage of the original" separate from "percentage increase", and state which quantities remain fixed.

Continue your physics journey
Videos, practice and support for your next step
www.physicsuk.co.uk

2 Choose the conductor geometry [7 marks]

The station needs a longer conductor. A and B are uniform round metal conductors of the same material at the same temperature. Their total conducting lengths and diameters are shown in Figure 1. Ignore insulation and connector resistance. Conductor A has resistance 0.800 Ω.

Use $R = \rho_e L/A_c$ and $A_c = \pi d^2/4$. Here ρ_e is resistivity and A_c is cross-sectional area. Form ratios before inserting values; use consistent units in each ratio.

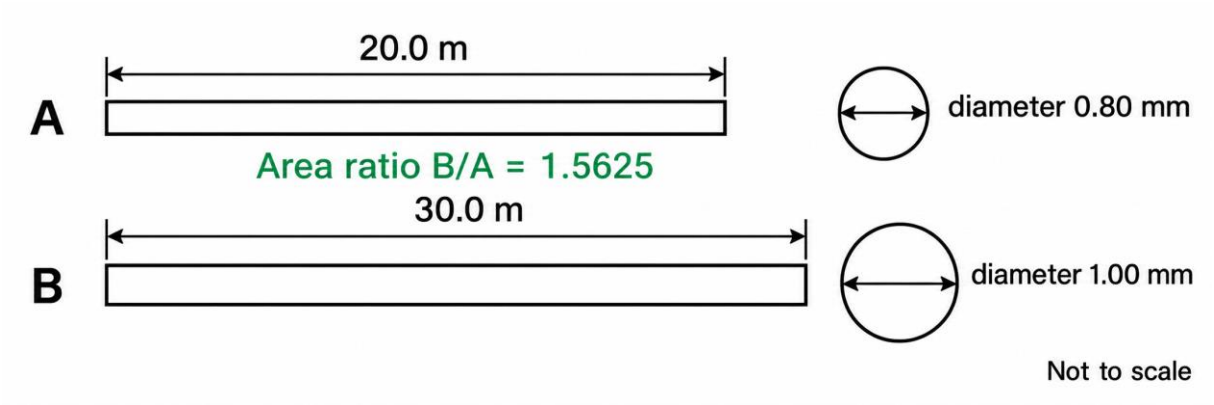


Figure 1 Candidate conductors and their circular cross-sections.

(a) Calculate A_{cB}/A_{cA}, the cross-sectional-area ratio. [1 mark]

(b) Obtain R_B/R_A using the length and diameter ratios. Show how the factors combine. [2 marks]

(c) Calculate the resistance of conductor B. [1 mark]

(d) A third conductor must be 1.50 times as long as A but have half its resistance. Find its required diameter as a factor of A's diameter, for the same material and temperature. [2 marks]

(e) Explain why both material and temperature were specified before cancelling ρ_e in part (b). [1 mark]

Worked answers and marking

(a) $A_{cB}/A_{cA} = (d_B/d_A)^2 = (1.00/0.80)^2 = 1.5625 \approx 1.56$. [1] The diameter ratio alone is not the area ratio.

(b) For equal resistivity, $R_B/R_A = (L_B/L_A)(d_A/d_B)^2$. [1] $R_B/R_A = (30.0/20.0)(0.80/1.00)^2 = 0.960$. [1] The area factor belongs in the denominator.

(c) $R_B = 0.960 \times 0.800 = 0.768\ \Omega$. [1]

(d) $0.50 = 1.50/(d/d_A)^2$, hence $(d/d_A)^2 = 3.00$. [1] $d/d_A = \sqrt{3} = 1.73$. [1] The required factor is $\sqrt{3}$, not 3.

(e) Both must give the same resistivity in the two conductors. The same material can have a different resistivity at a different temperature. [1] Accept equivalent wording linking both conditions to equal ρ_e.

Teaching prompt Do not confuse diameter with area. Use a labelled ratio before arithmetic; a longer but thicker conductor can still have a lower resistance.

Continue your physics journey
Videos, practice and support for your next step
www.physicsuk.co.uk

3 Compare mass and electrical losses [7 marks]

Continue with conductors A and B from Question 2. Their density ρ_m is the same. Each conductor is tested separately on a laboratory supply; these are controlled comparison tests, not predictions of the station's operating current.

For a uniform conductor, $m = \rho_m A_c L$. At fixed density, mass scales as Ld^2. Electrical heating power can be written as $P = I^2R$ or $P = V^2/R$. Choose the form that contains the quantity held constant.

(a) Calculate the mass ratio m_B/m_A. [1 mark]

(b) Explain why B has a slightly lower resistance than A, yet a much greater mass. Use the way length and area enter the two models. [2 marks]

(c) Each conductor carries a controlled current of 0.600 A. Find the ratio of heating powers P_B/P_A. [1 mark]

(d) Calculate the heating power in B during this constant-current test. [1 mark]

(e) In a new test, the same potential difference is maintained directly across each conductor. Calculate the new P_B/P_A and explain why the direction of the change is different. [2 marks]

Worked answers and marking

(a) $m_B/m_A = (L_B/L_A)(d_B/d_A)^2 = 1.50 \times 1.25^2 = 2.34375 \approx 2.34$. [1] No absolute density is needed because the material is unchanged.

(b) $R \propto L/A_c$, so B's area factor 1.5625 more than offsets its length factor 1.50 and R falls to $0.960R_A$. [1] Mass $\propto LA_c$, so both increases multiply and m rises to $2.34375m_A$. [1]

(c) With I fixed, $P_B/P_A = R_B/R_A = 0.960$. [1]

(d) $P_B = I^2R_B = 0.600^2 \times 0.768 = 0.27648$ W ≈ 0.276 W. [1] Accept a consistent value from the pupil's resistance.

(e) At fixed conductor p.d., $P_B/P_A = R_A/R_B = 1/0.960 = 1.04167 \approx 1.04$. [1] Lower R draws more current at fixed V, so heating power increases; in the earlier test I was fixed and $P = I^2R$ decreased. [1]

Teaching prompt "Same supply" is not enough to establish a scaling rule. The question specifies either the current through the conductor or the p.d. directly across it.

Continue your physics journey
Videos, practice and support for your next step
www.physicsuk.co.uk

4 Position an optical detector [7 marks]

An optical calibration source is used to check the station's detector. The supplied model is $s = CP_s/r^2$, where s is background-corrected detector signal, P_s is source power and r is distance from the emitting point. C stays constant for the same detector and orientation. Assume a point source, negligible absorption and no detector saturation.

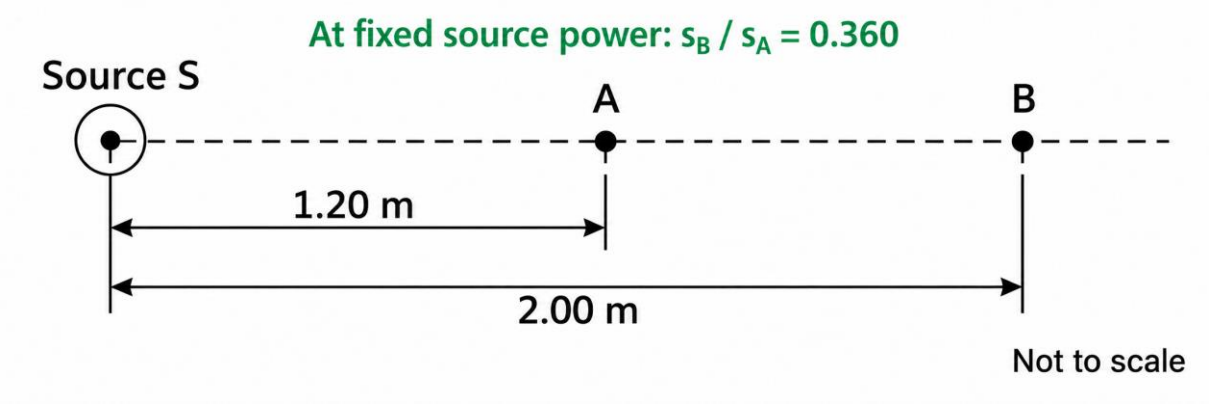


Figure 2 A and B are positions of the same detector, used one at a time.

(a) With source power fixed, calculate s_B/s_A for the two positions in Figure 2. [1 mark]

(b) Starting at A and keeping source power fixed, find how far the detector must move directly away from the source to halve its signal. [2 marks]

(c) The detector is now at B and source power is 1.50 times its original value. Compare this signal with the original signal at A by calculating their ratio. [2 marks]

(d) In a separate fixed-power test, distances of 1.00 m, 1.50 m and 2.00 m give background-corrected signals of 72, 32 and 18 arbitrary units. Test numerically whether these data fit the inverse-square model. [2 marks]

Worked answers and marking

(a) $s_B/s_A = (r_A/r_B)^2 = (1.20/2.00)^2 = 0.360$. [1]

(b) $0.50 = (1.20/r)^2$, so $r = 1.20\sqrt{2} = 1.697\ldots$ m. Correct new source distance. [1] Movement = 1.697… − 1.20 = 0.497 m (about 0.50 m). [1] Do not accept the source distance as the movement.

(c) $s_{newB}/s_{originalA} = (P_{new}/P_{original})(r_A/r_B)^2$. Correct combined comparison. [1] = $1.50 \times (1.20/2.00)^2 = 0.540$. [1]

(d) Calculate sr^2: $72 \times 1.00^2 = 72$; $32 \times 1.50^2 = 72$; $18 \times 2.00^2 = 72$ (in consistent signal–distance units). Constant product demonstrated. [1] The three data are consistent with $s \propto 1/r^2$ at fixed P_s and C. [1] Equivalent correct ratios accepted; agreement does not prove the law for all distances.

Teaching prompt Find the new distance from the source first, then subtract the original distance if asked for a movement. When both power and distance change, include both factors.

Continue your physics journey
Videos, practice and support for your next step
www.physicsuk.co.uk

5 Select a complete station design [10 marks]

The longer route is 1.50 times A's length. Candidate B is unchanged from Question 2. Candidate C has diameter 1.40 times A's diameter. Initially both use A's metal at A's temperature. Use fixed-current heating tests to compare cable losses. The table gives the battery energy and the measured total power draw for the proposed station; use these values directly for operating time.

Design limits: conductor mass ≤ $3.00m_A$; constant-current conductor loss ≤ $0.800P_A$; operating time ≥ 27.0 h. In the loss limit, P_A means the heating power of conductor A at the same test current. The upgraded station's measured power draw is separate from this conductor test.

Quantity	Original station	Upgraded station
Usable battery energy / W h	480	600
Constant total power draw / W	20.0	22.0
Operating time / h	24.0	To be determined

(a) Calculate R_C/R_A, showing how the simultaneous length and diameter changes combine. [2 marks]

(b) Calculate m_C/m_A and the percentage increase in conductor mass. [2 marks]

(c) Use an energy ratio and a power ratio to find the upgraded operating time from the original 24.0 h. Decide whether the time limit is met. [2 marks]

(d) For a separate C variant, a supplier uses a material with 0.80 times A's resistivity but the same C dimensions. Explain why the old R_C/R_A cannot simply be reused, and calculate the new ratio. [2 marks]

(e) Return to the original same-metal candidates B and C. Choose the conductor that meets both the mass and fixed-current loss limits. Justify your choice with both ratios. [2 marks]

Worked answers and marking

(a) $R_C/R_A = (L_C/L_A)(d_A/d_C)^2 = 1.50/1.40^2$. Correct combined ratio. [1] = 0.765306… ≈ 0.765. [1]

(b) $m_C/m_A = (L_C/L_A)(d_C/d_A)^2 = 1.50 \times 1.40^2 = 2.94$. [1] Increase = (2.94 − 1) × 100% = 194%. [1]

(c) $t_{up}/t_{old} = (E_{up}/E_{old})/(P_{up}/P_{old}) = 1.25/1.10 = 1.13636\ldots$. Correct combined ratio. [1] $t_{up} = 24.0 \times 1.13636\ldots$ = 27.27… h ≈ 27.3 h, meeting the 27.0 h limit. Result and decision. [1] Direct E/P calculation may verify the result, but the requested ratios must be shown.

(d) The resistivities no longer cancel; include $\rho_{e,new}/\rho_{e,A} = 0.80$. [1] $R_{newC}/R_A = 0.80 \times 1.50/1.40^2$ = 0.612245… ≈ 0.612. [1] Temperature conditions are assumed unchanged.

(e) Choose C: $m_C/m_A = 2.94 \le 3.00$. Correct mass check. [1] $P_C/P_A = R_C/R_A = 0.765\ldots \le 0.800$ at fixed current, so C also meets the loss limit. Correct loss check and choice. [1] B meets the mass limit (2.34) but fails the loss limit (0.960). Do not use the separate material variant from (d).

Teaching prompt A proportionality is a conditional statement. Check every constraint using the stated model and test conditions; reducing one cost does not automatically improve every other quantity.

Continue your physics journey
Videos, practice and support for your next step
www.physicsuk.co.uk

6 Transfer the ratio to a planetary probe [6 marks]

A probe studies a spherical planet of radius R = 3000 km. Its surface gravitational field strength has magnitude g_0 = 8.00 N kg^{-1}. Outside the planet use g = GM/r^2, where r is distance from its centre O. The planet's mass stays fixed. Figure 3 defines the probe's altitude h.

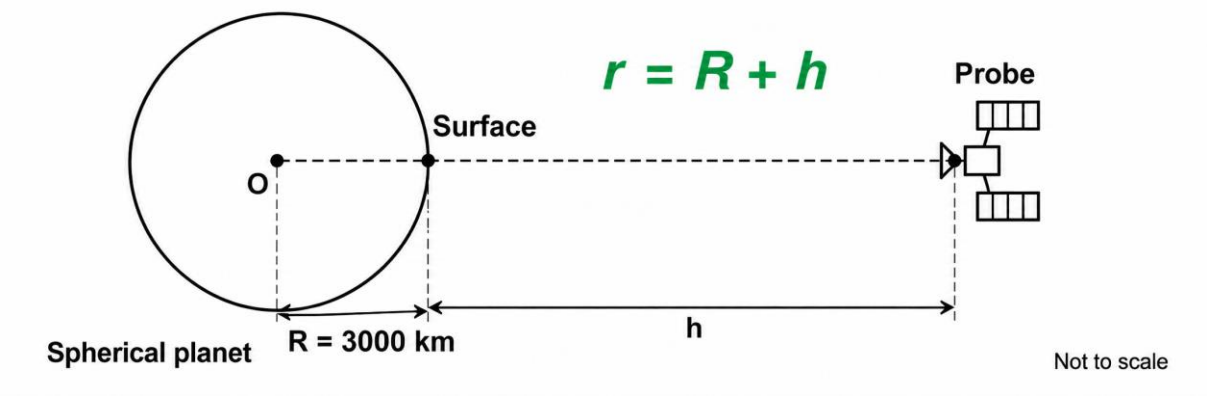


Figure 3 Altitude is measured from the surface, rather than from O.

(a) Calculate the gravitational field strength at an altitude of 1500 km, using a ratio to the surface value. [2 marks]

(b) Find the altitude at which the field strength is one ninth of its surface value. [2 marks]

(c) A pupil says "doubling altitude always quarters the field strength". Test the claim by comparing h = R with h = 2R, and explain the error. [2 marks]

Worked answers and marking

(a) r = R + h = 4500 km, so $g/g_0 = (3000/4500)^2 = 4/9$. Correct centre-distance ratio. [1] g = 8.00 × 4/9 = 3.56 N kg^{-1}. [1] Consistent km cancel; using h alone is incorrect.

(b) $1/9 = (R/r)^2$, hence r = 3R = 9000 km. Correct centre distance. [1] h = r − R = 2R = 6000 km. [1] Accept 6.00 × 10^6 m.

(c) The centre distances are 2R and 3R, so $g_{h=2R}/g_{h=R} = (2R/3R)^2 = 4/9$, not 1/4. Correct ratio. [1] The inverse-square law uses r = R + h; doubling altitude does not double distance from the centre. [1]

Teaching prompt For any inverse-square model, identify the reference point before forming the ratio. A useful sense check is that the field must decrease as the centre distance increases.

Final check State what is fixed, form a new/original ratio, apply every power, then check the direction of the change.

Continue your physics journey
Videos, practice and support for your next step
www.physicsuk.co.uk