Power laws and log–log graphs solutions

Matching worked solutions and teaching prompts | 40 marks

AQA A Level 7408 and OCR A A Level H556 | Mathematical skills

Skills and specification links

Use logarithms to find a power-law exponent and constant; interpret reference quantities and distinguish power laws from exponential change.
AQA maths MS 0.5, 2.2, 2.5, 3.1–3.4, 3.10–3.11; wire application: 3.5.1.3.
OCR A maths M0.5, M2.2, M2.5, M3.1–M3.4, M3.10–M3.11; wire application: 4.2.4(a)(i), (b).

A manufacturer tests fine-wire probes, an optical detector and a contact sensor. Use transformed graphs to test its models. Other than the wire equations, the physical models you need are supplied. Use a calculator and ruler. Show working; retain extra figures until the final answer.

1 Connect a power law to a straight line [4 marks]

For positive ratios $X = x/x_{ref}$ and $Y = y/y_{ref}$, a power law $Y = KX^n$ becomes $\log_{10} Y = n \log_{10} X + \log_{10} K$. Compare with $y = mx + c$: gradient = n; intercept $c = \log_{10} K$.
Using ln on both axes gives the same exponent, but the intercept is ln K. Recover K with the matching inverse: $K = 10^c$ for base 10, or $K = e^c$ for ln. Logarithms act on positive, dimensionless ratios.

(a) A calibration graph uses $\log_{10}$ on both axes and has intercept c = 0.40. Calculate K. [1 mark]
(b) A different calibration uses ln on both axes and has intercept c = 0.40. Calculate K. [1 mark]
(c) A detector has a straight log–log graph of response Y against separation X with gradient −1.5. State the power relationship and explain whether Y plotted directly against X would be a straight line. [2 marks]

Worked answers and marking
(a) $K = 10^{0.40} = 2.51$. [1] The intercept is $\log_{10} K$, not K itself.
(b) $K = e^{0.40} = 1.49$. [1] The inverse must match the logarithm base.
(c) Y is proportional to $X^{-1.5}$, or $Y = KX^{-1.5}$ for constant K. [1] The direct Y-against-X graph is non-linear: its gradient changes as X increases. [1] A straight transformed graph does not make Y directly proportional to X.

Teaching prompt Name the original variables and exponent. “A logarithmic relationship” does not identify the original power law.

2 Transform the fine-wire measurements [7 marks]

The probes use uniform circular wires of the same alloy, all of length 0.50 m and held at the same temperature. Let $X = d/d_{ref}$ and $Y = R/R_{ref}$, where $d_{ref} = 0.40$ mm and $R_{ref} = 1.00\ \Omega$. Diameter d and resistance R are measured below.

d / mm	0.20	0.25	0.32	0.40	0.50	0.63
R / Ω	8.10	5.00	3.15	1.98	1.30	0.810
$\log_{10} X$	−0.301		−0.097	0.000		0.197
$\log_{10} Y$		0.699		0.297	0.114	

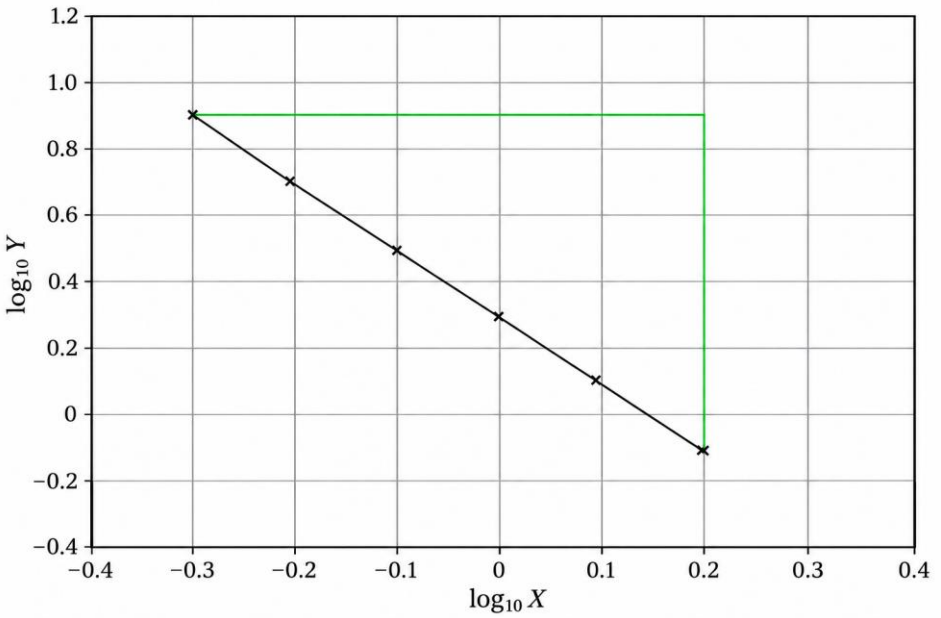


Figure 1 Plot $\log_{10} Y$ against $\log_{10} X$ on these linear scales.

(a) For the 0.25 mm wire, calculate X and Y. [1 mark]

(b) Complete the five blank log cells in the table to 3 decimal places. [2 marks]

(c) Plot all six points on Figure 1 and draw a straight line of best fit. [2 marks]

(d) For d = 0.20 mm, $\log_{10} X$ is negative. Explain why the diameter is still positive. [1 mark]

(e) State the diameter represented by $\log_{10} X = 0$. [1 mark]

Worked answers and marking

(a) X = 0.25/0.40 = 0.625; Y = 5.00/1.00 = 5.00. Both required. [1]

(b) Missing $\log_{10} X$: −0.204 and 0.097. Both correct. [1] Missing $\log_{10} Y$: 0.908, 0.498 and −0.092. All three correct. [1] Do not substitute ln values.

(c) All six points accurately plotted, within ±0.015 horizontally and ±0.03 vertically (about 2 mm at the printed size). [1] A sensible straight best-fit line through the trend, with points on both sides. [1] The teacher figure shows a representative fit, not point-to-point joins.

(d) X = 0.50 is positive but less than 1, so $\log_{10} X$ is negative; d remains +0.20 mm. [1]

(e) X = 1, so $d = d_{ref} = 0.40$ mm. [1] A zero logarithm does not mean zero diameter.

Teaching prompt The grid is linear in the transformed quantities. Convert to positive ratios before taking logarithms; a negative logarithm is valid.

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3 Recover the exponent and the physical constant [10 marks]

Use your line in Figure 1. The proposed model is $Y = KX^n$. For parts (b)–(e), use $n = -2.00$; retain your own intercept from the graph.

The dimensionless constant K depends on the chosen reference quantities. It is not automatically the SI constant k in $R = k\,d^{-2}$. In part (c), d must be expressed in metres.

(a) Draw a large gradient triangle on your line and use it to estimate n. [2 marks]

(b) Read the vertical intercept c and use it to find K. [2 marks]

(c) Starting with $R/R_{ref} = K(d/d_{ref})^{-2}$, find k in $R = k\,d^{-2}$. Give its SI unit. [2 marks]

(d) Use $R = \rho L/A_c$ and circular area $A_c = \pi d^2/4$ to explain why an exponent of −2 is expected. State what must remain constant. [2 marks]

(e) A new 0.48 mm wire must have resistance no greater than 1.50 Ω. Predict its resistance from your model and decide whether it meets this requirement. [2 marks]

Worked answers and marking

(a) Use a large triangle on the best-fit line, not two individual measured points. [1] Example endpoints (−0.30, 0.90) and (0.20, −0.10): $n \approx (-0.10 - 0.90)/(0.20 - (-0.30)) = -2.00$. [1] Accept about −2.15 to −1.85 with valid graphical working.

(b) $c \approx 0.30$ (accept about 0.27–0.33). [1] $K = 10^c \approx 2.0$ (about 1.86–2.14 over that intercept range). [1] K is dimensionless. Use the pupil's sensible intercept consistently.

(c) $R = K\,R_{ref}\,d_{ref}^2\,d^{-2}$, so $k = K\,R_{ref}\,d_{ref}^2$; use $d_{ref} = 4.0 \times 10^{-4}$ m. [1] For $K = 2.0$, $k = 3.2 \times 10^{-7}\ \Omega\ m^2$. [1] Accept consistent follow-through. K and k are different constants.

(d) $R = 4\rho L/(\pi d^2) = (4\rho L/\pi)d^{-2}$. Correct combination. [1] At fixed length L and resistivity ρ, R is proportional to d^{-2}; keeping alloy and temperature fixed keeps ρ constant. [1]

(e) $X = 0.48/0.40 = 1.20$; $R = (2.0)(1.00\ \Omega)(1.20)^{-2} = 1.39\ \Omega$. [1] It meets the ≤1.50 Ω requirement. [1] Accept consistent follow-through from K; within the accepted intercept range the prediction is still below 1.50 Ω. This is interpolation.

Teaching prompt The log–log gradient is the exponent. The intercept needs an inverse log. A physical constant also needs the reference quantities and their units.

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4 Test an inverse-square detector claim [7 marks]

An optical detector monitors a small source. Let $X = r/(1.00\ \text{m})$ and $Y = S/S_{ref}$, with reference response $S_{ref} = 1.00$ arbitrary unit. S is the background-corrected response. Its $\log_{10}$ Y-against-$\log_{10}$ X fit gives $n = -1.96 \pm 0.08$ and intercept $c = 1.30$. Measurements cover r = 0.50–2.00 m.

(a) Use the exponent's uncertainty range to assess $S \propto r^{-2}$ and $S \propto r^{-1}$. [2 marks]

(b) Use the fitted n and c to predict S at r = 1.20 m. [2 marks]

(c) The raw response is $S_{raw} = S + B$, where B is a positive constant background. Explain why B should be subtracted before taking logarithms, particularly at large r. [2 marks]

(d) Suggest another condition to control while changing r. Explain why it matters. [1 mark]

Worked answers and marking

(a) The exponent range is −2.04 to −1.88, which includes −2, so the result is consistent with inverse square. [1] It excludes −1, so inverse proportionality is not supported by the stated range. [1] Consistency does not prove the model at every separation.

(b) $K = 10^{1.30}$; $S = S_{ref} \times 10^{1.30}[r/(1.00\ \text{m})]^{-1.96}$. Correct recovery and model. [1] At 1.20 m, $S = 13.96 \approx 14.0$ arbitrary units. [1]

(c) Adding B means the raw response is not the same pure power law; an additive signal offset cannot be removed by a fixed correction after taking logarithms. [1] At large r the source signal is smaller and B forms a larger fraction, flattening the apparent decrease and biasing the exponent towards zero. [1]

(d) Keep source power constant so that changes in response are due to separation, rather than a changing source strength. [1] Accept fixed detector orientation with a valid sensitivity reason, or another relevant controlled condition with its purpose.

Teaching prompt Compare the proposed exponent with its uncertainty interval. Give evidence for consistency; do not claim that a fit proves a universal law.

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5 Choose between a power law and an exponential [7 marks]

The manufacturer obtains two sets of readings. For set P, X is a source–detector separation ratio. For set E, X = ℓ/(1.00 mm), where ℓ is the thickness of an optical attenuator. Each Y is a positive response divided by its own reference response. Consider $Y = KX^n$ and $Y = A\,e^{-bX}$.

X	1.00	2.00	3.00	4.00
Y: set P	96.0	24.0	10.7	6.00
Y: set E	96.0	48.0	24.0	12.0

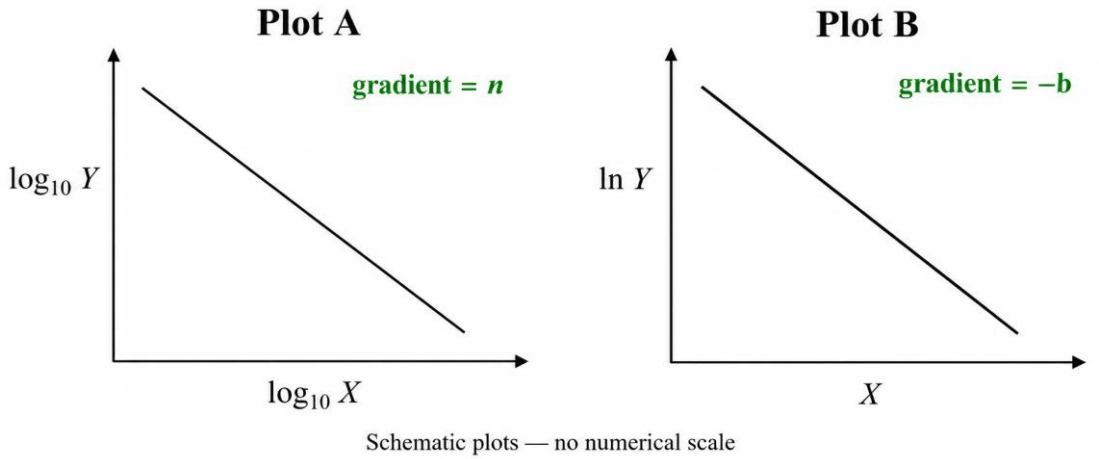


Figure 2 Two possible transformations; compare their horizontal axes.

(a) Use numerical ratios to identify which set is consistent with an inverse-square power law and which with exponential decrease. [2 marks]

(b) Match each set to the straight-line transformation in Figure 2. Justify each choice using the form of its equation. [2 marks]

(c) From set E, find the gradient of ln Y against X and hence b. Then find α in $Y = A\,e^{-\alpha\ell}$, in m^{-1}. [2 marks]

(d) Explain why a negative gradient alone cannot identify exponential decay. [1 mark]

Worked answers and marking

(a) P: doubling X from 1 to 2 (or 2 to 4) divides Y by 4, consistent with Y proportional to X^{-2}. [1] E: every increase of 1 in X halves Y, consistent with exponential decrease. [1] Allow the small rounding in P at X = 3.

(b) P uses Plot A: $\log_{10} Y = n \log_{10} X + \log_{10} K$, so both variables are transformed. [1] E uses Plot B: $\ln Y = \ln A - bX$, so X stays unlogged. [1]

(c) Gradient = ln(48/96)/(2 − 1) = ln 0.5 = −0.693, so b = +0.693. [1] Since X = ℓ/(10^{-3} m), α = b/(10^{-3} m) = 693 m^{-1}. [1]

(d) Both transformations can have negative gradients; the axis variables determine whether the model is a power law or an exponential. [1]

Teaching prompt A fixed multiplier for equal additions of X suggests an exponential. A fixed multiplier for equal factors of X suggests a power law. Always name the transformed axes.

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6 Transfer the method to a contact sensor [5 marks]

A contact sensor is calibrated with positive forces from 2.0 N to 16 N. Its tip deflects by δ. Define X = F/(1.00 N) and Y = δ/(1.00 mm). A log–log fit gives $\log_{10} Y = 0.67 \log_{10} X - 0.20$. This supplied model is valid over the measured range.

(a) Write an equation for δ in terms of F, including the reference quantities. [2 marks]

(b) Predict δ at F = 8.0 N. Decide whether it meets a maximum allowed deflection of 3.0 mm. [2 marks]

(c) Explain why a prediction for F = 64 N should not be treated as validated by this calibration. [1 mark]

Worked answers and marking

(a) n = 0.67 and $K = 10^{-0.20} = 0.631$. [1] $\delta = (0.631\ \text{mm})[F/(1.00\ \text{N})]^{0.67}$. [1] The exponent may be fractional; the power acts on the dimensionless force ratio.

(b) $\delta = 0.631 \times 8.0^{0.67}$ mm = 2.54 mm. [1] It is below 3.0 mm, so the sensor meets this requirement at 8.0 N. [1]

(c) 64 N is outside the measured 2–16 N range: it is extrapolation, and the response may change regime, so these data have not validated the model there. [1]

Teaching prompt Use the same recovery steps in an unfamiliar context: exponent, inverse-log constant, reference quantities, units and a check of the measured range.

Final check Identify the axes, find the exponent, recover the constant with the correct inverse log, then restore reference quantities and units.

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