

## Using In and log solutions

Matching worked solutions and teaching prompts | 42 core marks + 3 optional

AQA A Level 7408 and OCR A A Level H556 | Mathematical skills

### Skills practised

Use In, log and their inverses; apply logarithm laws; rearrange exponential equations; plot and interpret transformed data.

AQA maths MS 0.5, 2.5, 3.3, 3.4, 3.10, 3.11; applications 3.7.4.4 and 3.8.1.3 (selected content).

OCR A maths M0.5, M2.5, M3.3, M3.4, M3.10, M3.11; applications 6.1.3(b), (c), (e); 6.4.3(e)(i), (f)(i).

The maths is taught step by step. Physics equations and quantities are supplied. Use a scientific calculator where requested; otherwise reason from powers. Complete Questions 1–9 before the optional charging extension.

### 1 A logarithm tells you the power [4 marks]

A logarithm answers: what power of the base gives this number?

$10^3 = 1000$  means  $\log_{10}(1000) = 3$ . On this sheet, log means  $\log_{10}$ .

The number e is approximately 2.718. Natural logarithms use base e:  $e^x = a$  means  $\ln(a) = x$ .

Worked examples:  $\log(0.01) = -2$  because  $10^{-2} = 0.01$ ;  $\ln(e^2) = 2$ .

- (a) Write  $10^4 = 10000$  as an equivalent logarithm statement. [1 mark]
- (b) Find  $\log(0.001)$  without a calculator. Show the power of ten you use. [1 mark]
- (c) Find  $\ln(e^{-3})$  without a calculator. [1 mark]
- (d) A detector display uses the numbers  $\log(r/r_{\text{ref}})$ , where  $r$  and  $r_{\text{ref}}$  are count rates. The ratio  $r/r_{\text{ref}}$  rises from 10 to 1000. By how much does the displayed number increase? [1 mark]

#### Worked answers and marking

(a)  $\log(10000) = 4$ . [1]

(b)  $10^{-3} = 0.001$ , so  $\log(0.001) = -3$ . [1]

(c)  $\ln(e^{-3}) = -3$ . The logarithm returns the exponent. [1]

(d)  $\log(1000) - \log(10) = 3 - 1 = 2$ . [1] The signal ratio increases by a factor of 100, while its log increases by 2.

Teaching prompt A log value is an exponent. For log the base is 10; for ln the base is e. A negative exponent gives a positive fraction.



## 2 See what ln does [3 marks]

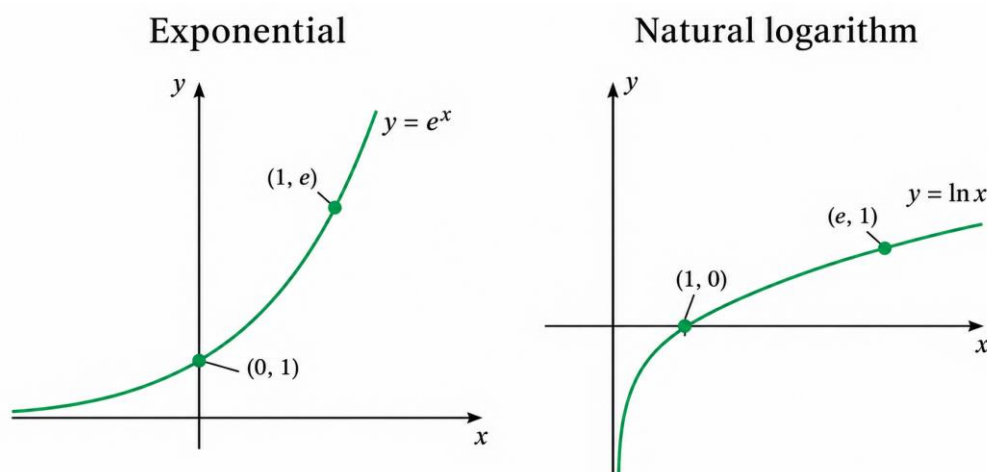


Figure 1 Schematic inverse functions. Use the labelled coordinates; do not measure the curves.

- (a) Use the labelled points in Figure 1 to state  $\ln(1)$  and  $\ln(e)$ . [1 mark]  
 (b) Which of these are valid real inputs to  $\ln$ :  $-0.2$ ,  $0$ ,  $0.2$  and  $2$ ? [1 mark]  
 (c) Explain why  $\ln(0.2)$  is negative even though  $0.2$  is positive. [1 mark]

### Worked answers and marking

- (a)  $\ln(1) = 0$  and  $\ln(e) = 1$ . Both required. [1] The input and output coordinates of the exponential are swapped.  
 (b)  $0.2$  and  $2$  only. [1] Real logarithms require a positive input.  
 (c) The power of  $e$  needed to produce a positive fraction below  $1$  is negative. [1] Accept:  $e$  to a negative power lies between  $0$  and  $1$ .

Teaching prompt The logarithm may be negative. That does not mean the input is negative. The curve exists only for  $x > 0$ .

## 3 Use and undo the calculator functions [4 marks]

Undo  $\ln$  with  $e^x$ ; undo  $\log$  with  $10^x$ . Use the appropriate second function on your calculator. Keep unrounded values until the final answer.

Worked example:  $\ln x = 0.70$  gives  $x = e^{0.70} = 2.01$ . Using  $1/\ln(0.70)$  would not undo  $\ln$ .

Table 1

| Expression  | Output | Expression   | Output |
|-------------|--------|--------------|--------|
| $\ln(0.35)$ |        | $\log(0.35)$ |        |
| $e^{1.40}$  |        | $10^{-1.40}$ |        |

- (a) Complete Table 1. Give each numerical output to three significant figures. [2 marks]  
 (b) Solve  $\ln x = -1.60$ . Give  $x$  to three significant figures. [1 mark]  
 (c) Solve  $\log x = 1.70$ . Give  $x$  to three significant figures. [1 mark]

### Worked answers and marking

- (a)  $\ln(0.35) = -1.05$ ;  $\log(0.35) = -0.456$ ;  $e^{1.40} = 4.06$ ;  $10^{-1.40} = 0.0398$ . Award [1] for the  $\ln/\log$  pair and [1] for the exponential pair.  
 (b)  $x = e^{-1.60} = 0.201896... = 0.202$ . [1]  
 (c)  $x = 10^{1.70} = 50.1187... = 50.1$ . [1]

Teaching prompt Match the inverse to its base:  $e^x$  undoes  $\ln$ ;  $10^x$  undoes  $\log$ .

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## 4 Use the laws of logarithms [3 marks]

For positive  $a$  and  $b$ :  $\ln(ab) = \ln a + \ln b$ ;  $\ln(a/b) = \ln a - \ln b$ ;  $\ln(a^n) = n \ln a$ .

The same laws work for log when every logarithm has the same base. There is no corresponding rule for splitting a sum or difference.

Worked example:  $\ln(3x) = \ln 3 + \ln x$  for  $x > 0$ .

(a) For  $x > 0$ , expand  $\ln(5x^2/2)$  into separate logarithms. [2 marks]

(b) A pupil writes  $\ln(8 + 2) = \ln 8 + \ln 2$ . Explain the mistake. [1 mark]

### Worked answers and marking

(a)  $\ln 5 + 2 \ln x - \ln 2$ . [1] for using product/quotient laws; [1] for the factor 2 from the power.

(b) Addition inside  $\ln$  cannot be split this way:  $\ln 8 + \ln 2$  equals  $\ln(8 \times 2) = \ln 16$ , not  $\ln 10$ . [1]

Teaching prompt The product rule is about multiplication inside the logarithm, not addition. The positive-input condition matters.

## 5 Release an unknown exponent [5 marks]

Worked example:  $20e^{-0.50x} = 5$ . First divide by 20:  $e^{-0.50x} = 0.25$ . Then take  $\ln$ :  $-0.50x = \ln(0.25)$ . Finally  $x = -\ln(0.25)/0.50 = 2.77$ .

Check: substituting  $x = 2.77$  gives approximately 5. Do not take  $\ln$  of a negative exponent by itself; take  $\ln$  of the whole positive exponential.

(a) Solve  $45e^{-0.30x} = 9$ . Show the isolated exponential and the logarithm step. [2 marks]

(b) Solve  $10^{0.40x} = 30$ . Use  $\log$  and show your working. [2 marks]

(c) For  $e^{-0.20x} = 0.60$ , a pupil obtains  $x = -2.55$ . Correct  $x$  and check the exponent's sign. [1 mark]

### Worked answers and marking

(a)  $e^{-0.30x} = 9/45 = 0.20$ ;  $-0.30x = \ln(0.20)$ . [1]  $x = -\ln(0.20)/0.30 = 5.36$ . [1]

(b)  $0.40x = \log(30)$ . [1]  $x = \log(30)/0.40 = 3.69$ . [1]

(c)  $x = -\ln(0.60)/0.20 = +2.55$ . [1] A positive  $x$  makes  $-0.20x$  negative, so the exponential is below 1. A negative  $x$  would make it exceed 1.

Teaching prompt Isolate the exponential before taking the logarithm. A negative  $\ln$  of a remaining fraction and the minus sign in the model must both be retained.

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## 6 Apply $\ln$ to a capacitor timing circuit [6 marks]

A hospital instrumentation team tests a capacitor timing circuit on a bench. The capacitor voltage falls as it discharges. All the physics you need is supplied below.

$V = V_0 e^{-t/\tau}$ , where  $V_0$  is the initial voltage,  $V$  the later voltage,  $t$  the elapsed time and  $\tau$  the time constant. Also  $\tau = RC$ , where  $R$  is resistance and  $C$  capacitance.  
 For a remaining fraction  $f = V/V_0$ :  $e^{-t/\tau} = f$ , so  $-t/\tau = \ln f$  and  $t = -\tau \ln f$ . The voltage ratio has no unit.  
 Worked example: for  $\tau = 4.0$  s and  $V/V_0 = 0.50$ ,  $t = -4.0 \ln(0.50) = 2.77$  s.

- (a) The test starts at  $V_0 = 12.0$  V and  $\tau = 10.0$  s. Find the time for  $V$  to fall to 3.00 V. Start with  $V/V_0$ , then  $\ln(V/V_0)$ , then  $t$ . [2 marks]
- (b) For the same circuit, find the time for the voltage to fall to 10.0% of its initial value. [1 mark]
- (c) A different circuit falls from 12.0 V to 4.80 V in 15.0 s. Determine its time constant  $\tau$ . [2 marks]
- (d) For a circuit with a constant  $\tau$ , compare the time for  $V$  to fall from 12 V to 6 V with the time to fall from 6 V to 3 V. Explain using the voltage ratios. [1 mark]

### Worked answers and marking

- (a)  $V/V_0 = 3.00/12.0 = 0.250$ , so  $\ln(V/V_0) = -1.38629\dots$  [1]  $t = -10.0 \times (-1.38629\dots) = 13.9$  s. [1]
- (b) Use  $f = 0.100$ , not 10:  $t = -10.0 \ln(0.100) = 23.0$  s. [1]
- (c)  $\ln(4.80/12.0) = -15.0/\tau$ , hence  $\tau = -15.0/\ln(0.400)$ . [1]  $\tau = 16.4$  s. [1]
- (d) The times are equal because each change has remaining fraction 0.5, and each interval is  $-\tau \ln(0.5)$ . [1]

Teaching prompt Use the remaining fraction, not the percentage lost. A falling positive voltage gives  $\ln(V/V_0) < 0$  but a positive elapsed time.



## 7 Transfer the method to radioactive decay [6 marks]

The same team monitors a radioactive reference sample at fixed detector geometry. A constant background adds to the detected count rate. The mean total count rates are 418 counts  $\text{s}^{-1}$  at  $t = 0$  and 118 counts  $\text{s}^{-1}$  at  $t = 12.0$  h. The background is 18 counts  $\text{s}^{-1}$ . Treat the background-corrected means as following the model below.

$r = r_0 e^{-\lambda t}$ , where  $r$  is the net count rate,  $r_0$  its initial net value and  $\lambda$  the decay constant. Use net count rate = total count rate – background count rate.

The net count rate is proportional to activity in this fixed setup; it is not the activity itself. Keep  $\lambda$  and  $t$  in matching time units.

Use the same steps: divide to get  $r/r_0$ , take  $\ln$ , then isolate the unknown. At the half-life  $T$ ,  $r/r_0 = 0.5$ .

- (a) Calculate the net count rate at each time. [1 mark]
- (b) Use your net count rates to determine  $\lambda$  in  $\text{h}^{-1}$ . Show the logarithm step. [2 marks]
- (c) Use  $0.5 = e^{-\lambda T}$  to find the half-life  $T$ . Keep your unrounded  $\lambda$  from part (b). [1 mark]
- (d) How long after the initial reading will the net count rate be 20.0% of its initial value? [2 marks]

### Worked answers and marking

- (a) Initially:  $418 - 18 = 400$  counts  $\text{s}^{-1}$ . At 12.0 h:  $118 - 18 = 100$  counts  $\text{s}^{-1}$ . Both required. [1]
- (b)  $\ln(100/400) = -\lambda \times 12.0$ . [1]  $\lambda = -\ln(0.25)/12.0 = 0.1155\dots \text{h}^{-1}$ , or  $0.116 \text{ h}^{-1}$  to three significant figures. [1]
- (c)  $\ln(0.5) = -\lambda T$ ;  $T = -\ln(0.5)/\lambda = \ln 2/\lambda = 6.00$  h. [1]
- (d)  $e^{-\lambda t} = 0.200$ , so  $t = -\ln(0.200)/\lambda$ . [1] Using  $\lambda = \ln(4)/12.0$  gives  $t = 13.9$  h. [1]

Teaching prompt Subtract background from both readings before taking the ratio. Do not use  $118/418$ . Do not mix hours with a decay constant in  $\text{s}^{-1}$ .



## 8 Turn exponential data into a straight line [5 marks]

Figure 2 compares a remaining fraction  $f$  with its natural logarithm. These are schematic shapes. The right-hand plot uses ordinary linear axes; negative  $\ln$  values lie below zero.

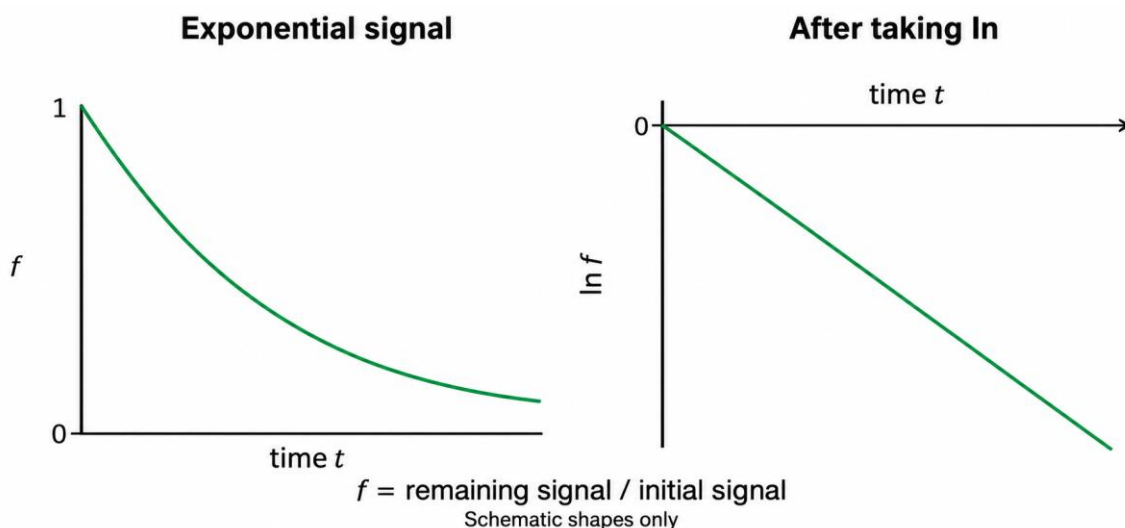


Figure 2 The logarithm of an exponentially decreasing fraction decreases linearly.

Taking  $\ln$  undoes  $e$  to a power. For  $f = e^{-kt}$ ,  $\ln f = -kt$ . This has the straight-line form  $y = mx + c$ , with  $y = \ln f$ ,  $x = t$ ,  $m = -k$  and  $c = 0$ .

A separate capacitor test gives Table 2. Its model is  $V = V_0 e^{-t/\tau}$ , with  $V_0 = 9.00 \text{ V}$ . Its  $\tau$  is unknown. Use this table, not Figure 2, for numerical work.

Scale check: a logarithmic axis can place the labels 0.001, 0.01, 0.1, 1 at equal distances. Their calculated log values  $-3$ ,  $-2$ ,  $-1$ ,  $0$  go on an ordinary linear axis.

Table 2

| $t / \text{s}$ | $V / \text{V}$ | $\ln(V/V_0)$ |
|----------------|----------------|--------------|
| 0              | 9.00           | 0.000        |
| 10             | 5.46           |              |
| 20             | 3.31           |              |
| 30             | 2.01           |              |
| 40             | 1.22           |              |

(a) Complete the four missing  $\ln(V/V_0)$  values in Table 2 to three decimal places. [2 marks]

(b) Starting from the capacitor model, obtain an equation for  $\ln(V/V_0)$  against  $t$ . Identify its gradient in terms of  $\tau$ . [2 marks]

(c) Describe how  $V$  and  $\ln(V/V_0)$  each change as time increases. [1 mark]

### Worked answers and marking

(a) At  $t = 10, 20, 30$  and  $40 \text{ s}$  the values are  $-0.500$ ,  $-1.000$ ,  $-1.499$  and  $-1.998$ . [1] for the first two correct values and [1] for the last two. Keep unrounded values for plotting if desired.

(b) Divide by  $V_0$ , then take  $\ln$ :  $\ln(V/V_0) = -t/\tau$ . [1] Comparing with  $y = mt + c$  gives  $m = -1/\tau$  and  $c = 0$ . [1]

(c)  $V$  decreases non-linearly (exponentially), while  $\ln(V/V_0)$  decreases linearly. Both statements required. [1]

Teaching prompt The plotted  $y$  values are negative because  $V/V_0$  is between 0 and 1. The axis is linear in  $\ln(V/V_0)$ , not a raw-voltage axis.

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## 9 Plot and interpret the transformed data [6 marks]

Use your completed Table 2 for Figure 3. The printed axes are ordinary linear axes: each horizontal grid division is 5 s and each downward vertical grid division is a decrease of 0.25 in  $\ln(V/V_0)$ .

(a) Plot the five points from Table 2 on Figure 3 and draw a straight line of best fit. [2 marks]

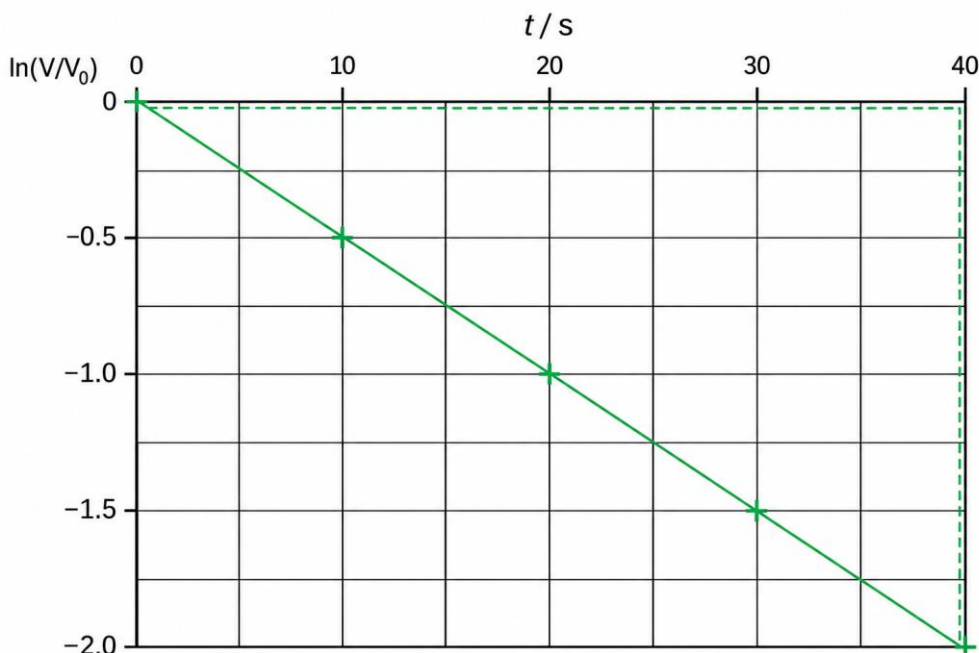


Figure 3 Representative best-fit line and gradient triangle.

(b) Use a large gradient triangle to determine the gradient, including its unit. Hence determine  $\tau$ . [2 marks]

(c) Another pupil plots  $\log(V/V_0)$  instead of  $\ln(V/V_0)$  and obtains gradient  $g = -0.0217 \text{ s}^{-1}$ . They calculate  $\tau = -1/g = 46.1 \text{ s}$ . Explain the error and find the correct  $\tau$ . Use  $\ln x \approx 2.303 \log x$ . [2 marks]

### Worked answers and marking

(a) Plot (0,0), (10,−0.500), (20,−1.000), (30,−1.499), (40,−1.998). [1] for accurate points (within about 1 s horizontally and 0.05 vertically) and [1] for a sensible straight best-fit line spanning the data. The figure uses a representative line; tiny departures from grid intersections are below print resolution.

(b) Using endpoints on the line:  $m = (-2.00 - 0)/(40.0 - 0) = -0.0500 \text{ s}^{-1}$ . [1]  $\tau = -1/m = 20.0 \text{ s}$ . [1] Accept  $m$  from  $-0.052$  to  $-0.048 \text{ s}^{-1}$  and  $\tau$  consistent with the pupil line (approximately 19.2–20.8 s).

(c) The gradient  $-1/\tau$  belongs to the  $\ln$  plot. Here  $m \approx 2.303g$ , so  $\tau \approx -1/(2.303g)$ . [1]  $\tau \approx -1/[2.303 \times (-0.0217)] = 20.0 \text{ s}$ . [1]

Teaching prompt Take the change in the plotted quantity, not a change in voltage. A decreasing straight line has a negative gradient; the time constant is positive.

## 10 Optional extension on charging [3 marks]

Allow about 5 extra minutes. During charging the supplied model is  $V = V_{\max}(1 - e^{-t/\tau})$ , where  $V_{\max}$  is the final steady voltage.

(a) Rearrange the charging model to make  $t$  the subject. Isolate the exponential before taking  $\ln$ . [2 marks]

(b) For  $\tau = 8.00 \text{ s}$ , calculate the time to reach 75.0% of the final voltage. [1 mark]

### Worked answers and marking

(a)  $e^{-t/\tau} = 1 - V/V_{\max}$ . [1] Therefore  $t = -\tau \ln(1 - V/V_{\max})$ . [1] For a finite positive charging time the voltage fraction lies between 0 and 1.

(b) The exponential equals  $1 - 0.750 = 0.250$ .  $t = -8.00 \ln(0.250) = 11.1 \text{ s}$ . [1]

Teaching prompt For charging, take  $\ln$  of the fraction still to be gained:  $1 - V/V_{\max}$ . It is not  $\ln(1) - \ln(V/V_{\max})$ .

Review check Identify the base, undo its logarithm, isolate the exponential and check the sign.

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