Gradients tangents and areas solutions

Matching worked solutions and teaching prompts | 39 marks

AQA A Level 7408 and OCR A A Level H556 | Mathematical skills

Skills practised

Interpret axes and units; use secants and tangents; calculate signed areas and estimate curved areas.
AQA maths MS 0.1, 2.3, 3.1, 3.5–3.8, 4.3; selected applications 3.4.1.3, 3.4.1.6–3.4.1.7; 3.7.4.1, 3.7.4.3.
OCR A maths M0.1, M2.3, M3.1, M3.5–M3.8, M4.3; selected applications 3.1.1(a)–(d); 3.5.1(b)–(e); 6.1.1(a); 6.1.2(a)–(c).

A battery-powered inspection shuttle is tested on a straight track. Study its motion, a separate braking pulse and a capacitor buffer for its sensor electronics. Each test has its own time origin. Use a ruler and calculator; show graphical working and give units.

1 Read the axes first [3 marks]

A gradient is change in the vertical quantity divided by change in the horizontal quantity. Its unit is vertical unit ÷ horizontal unit. Example: a temperature–time gradient has unit K s^{-1}; it is a temperature-change rate, not an acceleration.
An area has unit vertical unit × horizontal unit. Use a secant between two curve points for an average rate, and a tangent at one point for an instantaneous rate. Check what the units mean before calculating.

(a) State the physical quantity and SI unit obtained from the gradient of displacement x against time t. [1 mark]
(b) Give the unit of an area on a force–time graph, and write it in base SI units. [1 mark]
(c) A technician wants the shuttle's acceleration at one instant from a curved velocity–time graph. State the graphical method and the unit of the result. [1 mark]

Worked answers and marking
(a) Velocity, in m s^{-1}. Both quantity and unit required. [1]
(b) N s = kg m s^{-1}. Both equivalent forms required. [1] These are momentum units, not energy units.
(c) Draw a tangent at the required time and calculate its gradient; unit m s^{-2}. [1] A secant gives an average over an interval.

Teaching prompt An operation has no fixed physical meaning without its axes. First identify the graph quantities, then check the resulting units.

2 Compare average and instantaneous velocity [5 marks]

Trial A records the shuttle's displacement x from its starting point. Positive x is along the track. Use Figure 1; the curve represents the measured motion.

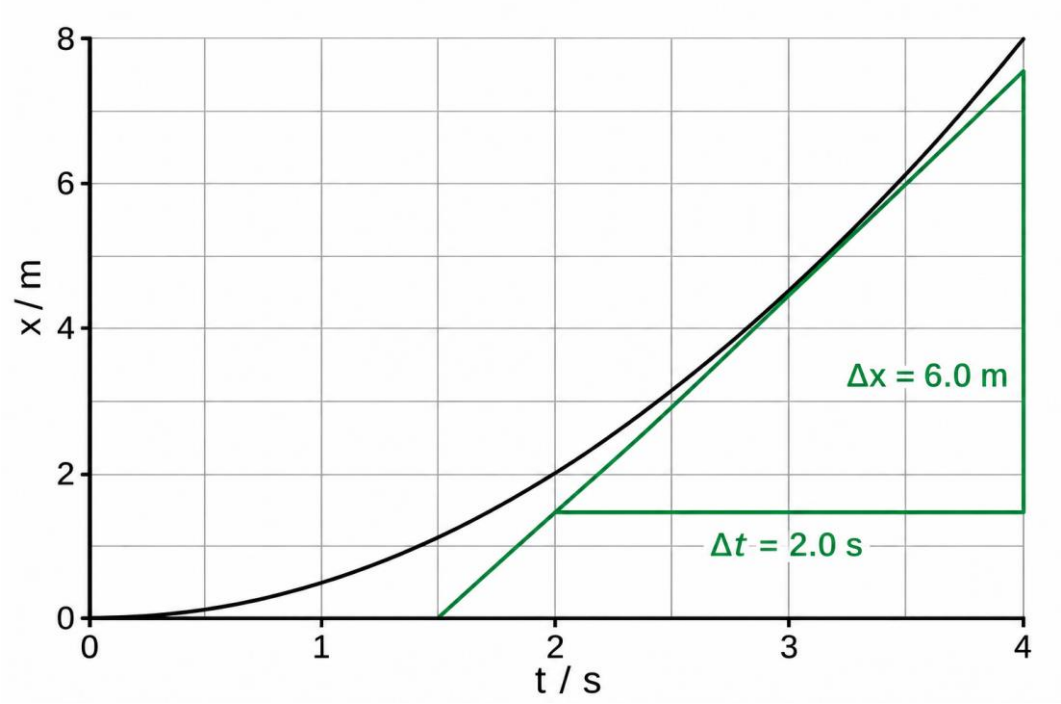


Figure 1 Displacement during the first acceleration trial.

(a) Calculate the average velocity from t = 0 to t = 4.0 s. [1 mark]

(b) Draw a tangent to the curve at t = 3.0 s. [1 mark]

(c) Use a large gradient triangle on your tangent to find the instantaneous velocity at t = 3.0 s. [2 marks]

(d) Explain why the instantaneous velocity in part (c) is greater than the average in part (a). [1 mark]

Worked answers and marking

(a) Average velocity = $\Delta x/\Delta t = (8.0 - 0)/(4.0 - 0) = 2.0$ m s^{-1}. [1]

(b) A straight tangent touching at about (3.0 s, 4.5 m), following the local curve direction without cutting across it nearby. [1] The teacher image shows a representative tangent.

(c) Example tangent points: (2.0 s, 1.5 m) and (4.0 s, 7.5 m). A large triangle using the tangent, rather than two points on the curve. [1] $v = (7.5 - 1.5)/(4.0 - 2.0) = 3.0$ m s^{-1}. [1] Accept about 2.8–3.2 m s^{-1} with a sensible tangent and working. Do not use x/t.

(d) The displacement gradient increases during the trial, so the shuttle's velocity increases; the earlier, lower velocities reduce the average over 0–4 s. [1] Accept an equivalent comparison of the tangent and secant gradients.

Teaching prompt A curve point gives displacement, not velocity. A tangent uses two points on the tangent line; they need not be points on the original curve.

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3 Keep the sign when finding an area [11 marks]

Trial B is a separate manoeuvre. Figure 2 shows signed velocity v. Positive velocity is forward along the track; negative velocity is backward.

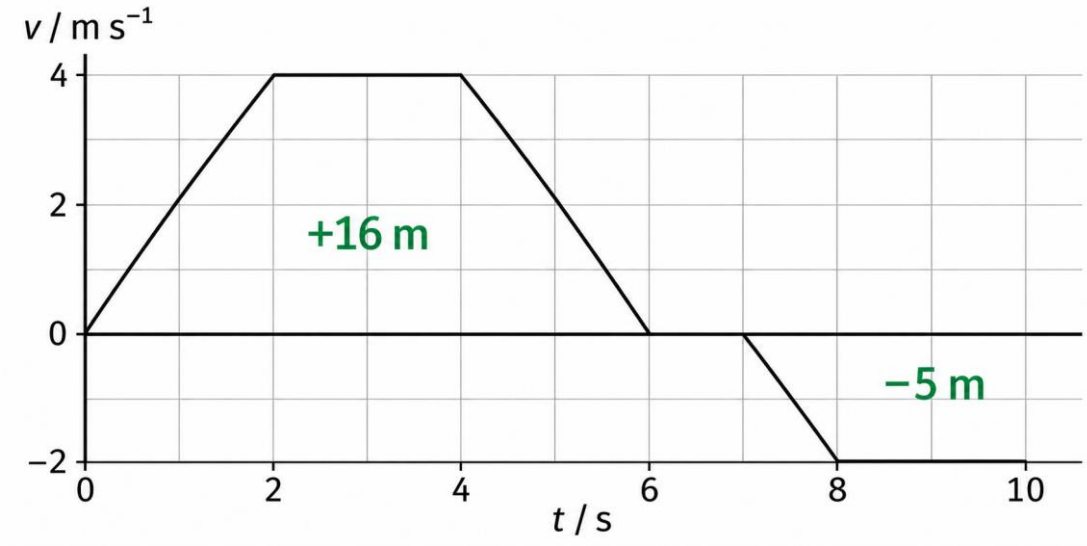


Figure 2 Forward travel, a pause and reverse travel.

(a) Use areas to find the displacement from t = 0 to t = 6.0 s. [2 marks]

(b) Find the signed displacement from t = 6.0 s to t = 10.0 s. [2 marks]

(c) Hence find both the total displacement and total distance travelled from t = 0 to t = 10.0 s. [2 marks]

(d) Describe the motion from t = 6.0 s to t = 10.0 s. Include the pause, direction and how speed changes in each moving section. [3 marks]

(e) Calculate the acceleration from t = 7.0 s to t = 8.0 s. Explain why its negative sign does not mean that the shuttle is slowing down. [2 marks]

Worked answers and marking

(a) Positive areas: ½ × 2 × 4 = 4 m; 2 × 4 = 8 m; ½ × 2 × 4 = 4 m. Correct decomposition into the two triangles and rectangle. [1] Displacement = 4 + 8 + 4 = +16 m. [1] Equivalent trapezium methods accepted.

(b) The zero-velocity section contributes zero. Reverse-travel area magnitude = ½ × 1 × 2 + 2 × 2 = 5 m. [1] Signed displacement = −5 m because v is negative. [1]

(c) Total displacement = 16 − 5 = +11 m. [1] Total distance = 16 + 5 = 21 m. [1] Distance adds the magnitudes of the areas.

(d) 6–7 s: at rest. [1] 7–8 s: moving backward with speed increasing linearly from 0 to 2 m s^{-1}. [1] 8–10 s: moving backward at constant speed 2 m s^{-1}. [1] Velocity is negative; speed is never negative.

(e) $a = \Delta v/\Delta t = (-2 - 0)/(8 - 7) = -2.0$ m s^{-2}. [1] Velocity becomes more negative while its magnitude increases, so the shuttle is speeding up backward. [1]

Teaching prompt Signed area gives displacement. For distance, add magnitudes. Negative velocity indicates direction; acceleration and speed need separate interpretations.

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4 Estimate the effect of a braking pulse [8 marks]

In trial C, the 10.0 kg shuttle starts at +6.00 m s^{-1}. Figure 3 shows the resultant horizontal force during a 1.00 s braking pulse. Forward is positive. The table gives the exact force readings to use in your estimate.

Signed trapezium contribution = ½(F_1 + F_2)Δt. Use four intervals of width 0.25 s. Retain the force signs.

t / s	0	0.25	0.50	0.75	1.00
F / N	−120	−67.5	−30.0	−7.5	0

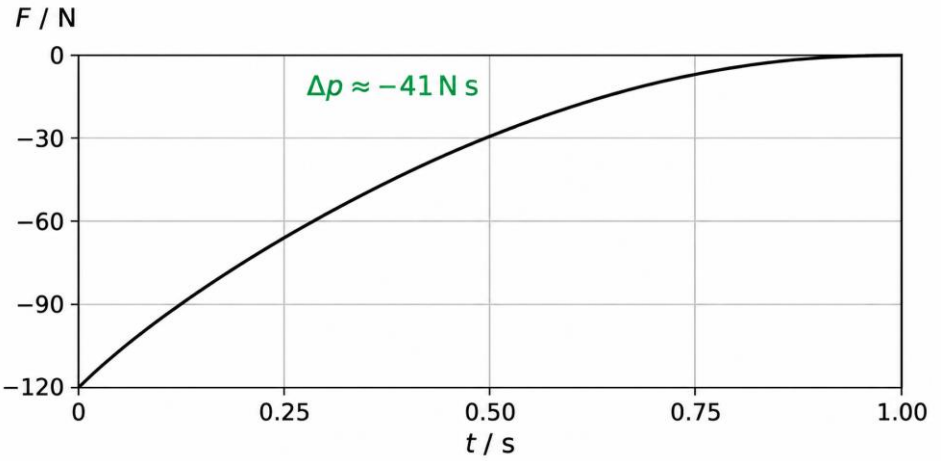


Figure 3 Resultant braking force; use the table values for numerical areas.

(a) State what the signed area between this force curve and F = 0 represents, with its unit. [1 mark]

(b) Use four trapezia and the table to estimate the signed impulse over 0–1.00 s. [2 marks]

(c) Calculate the initial momentum, then use your impulse to estimate the final momentum. [2 marks]

(d) Estimate the final velocity, including its direction. [1 mark]

(e) A pupil uses the initial force for the whole second. Calculate this rectangular impulse estimate and explain why it overestimates the magnitude of the braking effect. [2 marks]

Worked answers and marking

(a) Resultant impulse, equal to the shuttle's momentum change; unit N s or kg m s^{-1}. [1]

(b) J ≈ 0.25/2 × [−120 + 2(−67.5 − 30.0 − 7.5) + 0]. Correct signed trapezium sum. [1] J ≈ −41.25 N s ≈ −41 N s. [1] Equivalent sum of −23.4375, −12.1875, −4.6875 and −0.9375 N s accepted.

(c) p_i = mu = 10.0 × 6.00 = +60.0 kg m s^{-1}. [1] p_f = p_i + J = 60.0 − 41.25 = +18.75 kg m s^{-1}. [1] Accept consistent follow-through from a valid estimate.

(d) v_f = p_f/m = 18.75/10.0 = +1.875 m s^{-1} ≈ 1.9 m s^{-1}, forward. [1] Accept consistent follow-through.

(e) J_{rect} = −120 × 1.00 = −120 N s. [1] The force magnitude decreases during the pulse, so assuming the initial maximum magnitude throughout includes too much negative area. [1] It would wrongly predict reversal for these data.

Teaching prompt This is resultant force, so its impulse is the net momentum change. Keep the initial momentum and the signed change separate; braking does not necessarily bring an object to rest.

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5 Transfer area reasoning to electrical energy [8 marks]

A capacitor buffer supplies brief pulses to the shuttle's sensor electronics. It is separate from the main traction-energy store. Figure 4 shows voltage V against charge Q for an ideal capacitor of constant capacitance C. Stored energy gained equals the area under V against Q. Use C = Q/V and 1 mC = 10^{-3} C.

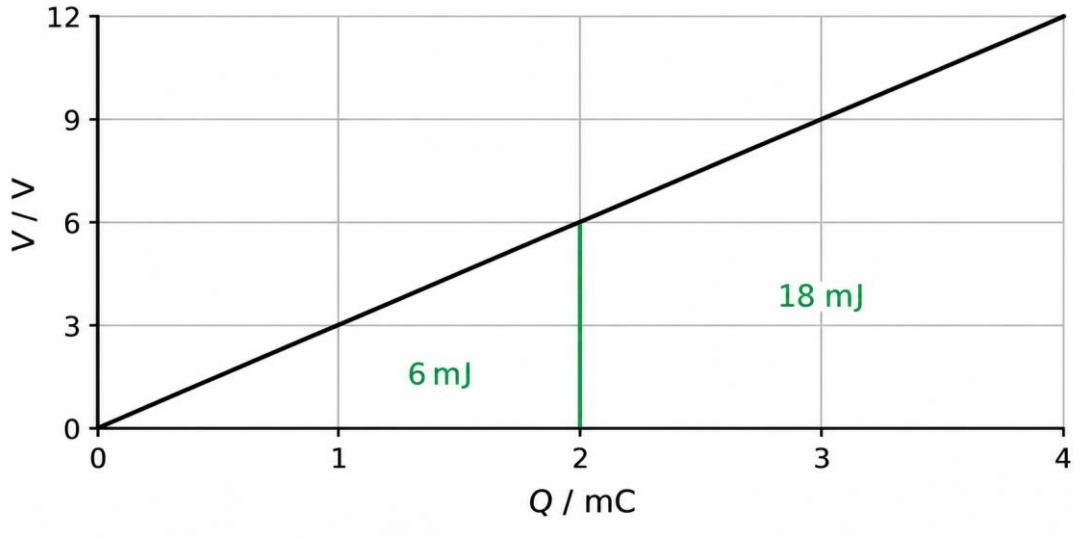


Figure 4 Voltage against charge for the sensor capacitor.

(a) Find the energy stored when the initially uncharged capacitor reaches Q = 4.0 mC. Give your answer in joules. [2 marks]

(b) Compare the energy gained over 0–2.0 mC with that gained over 2.0–4.0 mC. Calculate both values and state their ratio, later interval to earlier interval. [2 marks]

(c) A second capacitor has a V-against-Q line through the origin with twice this gradient. Find its capacitance as a factor of the first capacitor's, and its stored energy factor at the same final charge. [2 marks]

(d) Swap the axes to Q against V. State the gradient's meaning. For this linear capacitor, show whether the zero-to-final triangular area in SI units still equals stored energy. [2 marks]

Worked answers and marking

(a) E = ½ × 4.0 × 12 = 24 in mC V, so 24 mJ. Correct triangular area. [1] E = 24 × 10^{-3} = 0.024 J. [1]

(b) First interval: ½ × 2.0 × 6.0 = 6.0 mJ. Later interval: ½(6.0 + 12.0) × 2.0 = 18 mJ. Both correct areas. [1] Later/earlier energy = 18/6 = 3.0. [1] Equal charge intervals do not imply equal energy transfers.

(c) For V against Q, gradient = 1/C, so doubling the gradient halves C: C_2/C_1 = 0.50. [1] At the same Q its voltage and triangular area double, so E_2/E_1 = 2.0. [1] Equivalent use of E = $Q^2/(2C)$ accepted.

(d) The Q-against-V gradient is capacitance C. [1] Yes: the zero-to-final triangle still has area ½QV = 0.024 J. [1] This equality uses the stated linear constant-C relation and a zero starting charge; it is not a general rule that swapping arbitrary axes preserves an area's physical interpretation.

Teaching prompt Read the axis order. V against Q has gradient 1/C; Q against V has gradient C. Both zero-origin triangles give ½QV for this ideal linear capacitor.

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6 Explain and check a graph result [4 marks]

(a) A technician means that positive velocity increases at a constant rate, but writes “the velocity is accelerating”. Rewrite this precisely, including what happens to acceleration. [1 mark]

(b) For the smooth braking curve, explain why using more, narrower trapezia would usually improve the impulse estimate. Explain why the result would still be an estimate. [2 marks]

(c) A pupil calls the area under a force-against-velocity graph “energy” because it has the same triangular shape as Figure 4. Use the axis units to test this claim. [1 mark]

Worked answers and marking

(a) Velocity increases linearly with time; acceleration is constant and positive. [1] Accept equivalent precise wording.

(b) Shorter straight segments generally follow the smooth curve more closely, reducing the gaps between each chord and the curve. [1] The force can still curve between the sampled readings, so the trapezia do not give the exact area. [1]

(c) Its area unit is $\mathrm{N \times m\ s^{-1}} = \mathrm{W}$, not J, so the area cannot be labelled energy. [1] Shape alone is not a physical interpretation; this does not assert a general formula for power from that area.

Teaching prompt Name the quantity, its change and the relevant mathematical operation. Units can reject an interpretation, but a shared unit or a familiar shape alone does not prove a physical meaning.

Final check Read the axes and units, choose tangent or area, keep signs, then check the physical meaning.

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